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RESEARCH ARTICLE

On instability of zero solution of some types of forth order

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Abstract

The purpose of this paper is to study the instability of the zero solution of some type of nonlinear delay differential equations of fourth order by using the Lyapunov-Krasovskii functional approach; we obtain some conditions of instability of solution of such equation.

INTRODUCTION

In 2000, Ezeilo [5], proved two instability theorem for the fourth order nonlinear differential equation without delay

$$x^{(4)} + a_1\dddot{x} + g(x,\dot{x},\ddot{x},\dddot{x})\ddot{x} + h(x)\dot{x} + f(x,\dot{x},\ddot{x},\dddot{x}) = 0$$

And

$$x^{(4)} + p(\dddot{x},\ddot{x}) + q(x,\dot{x},\ddot{x},\dddot{x})\ddot{x} + a_3\dot{x} + a_4x = 0$$

While Tunc in 2009, [2], gave two instability theorems to fourth order nonlinear differential equation with a variable delay $\tau(t)$

$$x^{(4)}(t) + a_1\dddot{x}(t) + g\left(x\big(t-\tau(t)\big),\cdots,\dddot{x}\big(t-\tau(t)\big)\right)\ddot{x} + h(x(t))\dot{x}(t) + f\left(x\big(t-\tau(t)\big),\cdots,\dddot{x}\big(t-\tau(t)\big)\right)x(t) = 0$$

And

$$x^{(4)}(t) + p\big(\dddot{x}(t),\ddot{x}(t)\big) + q\left(x\big(t-\tau(t)\big),\cdots,\dddot{x}\big(t-\tau(t)\big)\right)\ddot{x}(t) + a_3\dot{x}(t) + a_4x(t) = 0$$

Also Tunc [4], in 2011 introduced new instability theorems to fourth order nonlinear differential equation with constant delay of the form

$$x^{(4)} + \psi(\ddot{x})\dddot{x} + g\left(x, x(t-\tau),\cdots,x^{(3)},x^{(3)}(t-\tau)\right)\ddot{x} + \theta(\dot{x}) + f\big(x(t-\tau)\big) = 0$$

And in (2013) Tunc [3], gave sufficient condition for instability of the zero solution of a vector nonlinear differential equation of fourth order with constant delay of the form:

$$X^{(4)} + A\dddot{X} + H\big(\dot{X}\big)\ddot{X} + G(X)\dot{X} + F\big(X(t-\tau)\big) = 0$$

In this paper, we study a two types of fourth order nonlinear delay differential equation with a variable delay, the first type is of the form

$$X^{(4)} + \Phi(\ddot{X})\dddot{X} + \Omega(X, \dot{X}, \ddot{X}, \dddot{X})\ddot{X} + \psi(\ddot{X}) + \Theta\left(\dot{X}(t - \tau(t))\right) = 0 \cdots (1)$$

Where $X \in \mathbb{R}^n$, $\tau(t)$ is a variables delay, Φ, Ω are continuous $n \times n$-symmetric matrix functions, $\psi: \mathbb{R}^n \to \mathbb{R}^n$, $\Theta: \mathbb{R}^n \to \mathbb{R}^n$,with $\psi(0) = \Theta(0) = 0$ are continuous for all $X \in \mathbb{R}^n$
And the second type is

$$X^{(4)} + F\left(X(t), X(t - \tau(t)), \cdots, \dddot{X}(t), \dddot{X}(t - \tau(t))\right)\dddot{X}(t) +$$
$$G\left(X(t), X(t - \tau(t)), \cdots, \dddot{X}(t), \dddot{X}(t - \tau(t))\right)\ddot{X}(t) + H\left(\dot{X}(t - \tau(t))\right) + \Psi\left(\ddot{X}(t)\right) + AX(t) = 0 \cdots (2)$$

Where $X \in \mathbb{R}^n$, $\tau(t)$ is a variables delay, A is a constant symmetric $n \times n$ matrices, F and G are continuous $n \times n$-symmetric matrix functions, $H: \mathbb{R}^n \to \mathbb{R}^n$, $\Psi: \mathbb{R}^n \to \mathbb{R}^n$,with $H(0) = \Psi(0) = 0$ are continuous for all $X \in \mathbb{R}^n$ and the primes in(1,2) denote differentiation with respect to $t, t \in \mathbb{R}^+$. And we obtain some conditions of instability of the zero solution using the Lyapunov- krasovskii functional approach.
First we write equations (1) and (2) in systems form as follows:

$$\dot{X} = Y, \dot{Y} = Z, \dot{Z} = W$$
$$X^{(4)} = \dot{W} = -\Phi(Z)W - \Omega(X, Y, Z, W)Z - \psi(Z) - \Theta(Y) + \int_{t-\tau(t)}^{t} J_\Theta(Y(s))Z(s)ds \cdots (3)$$

And

$$\begin{aligned} X^{(4)} = \dot{W} = &-F\left(X(t), X(t - \tau(t)), Y(t), Y(t - \tau(t)), \cdots, W(t), W(t - \tau(t))\right)W \\ &- G\left(X(t), X(t - \tau(t)), Y(t), Y(t - \tau(t)), \cdots, W(t), W(t - \tau(t))\right)Z - H(Y) - \Psi(Z) - AX \\ &+ \int_{t-\tau(t)}^{t} J_H(Y(s))Z(s)ds \cdots (4) \end{aligned}$$

Where $J_\Theta(Y), J_\psi(Z), J_\Psi(Z)$ and $J_H(Y)$, denote the linear operator form $\Theta(Y), \psi(Z), \Psi(Z)$ and $H(Y)$ to $J_\Theta(Y) = \left(\frac{\partial \Theta_i}{\partial y_j}\right)$, $J_\psi(Z) = \left(\frac{\partial \psi_i}{\partial z_j}\right)$, $J_\Psi(Z) = \left(\frac{\partial \Psi_i}{\partial z_j}\right)$, and $J_H(Y) = \left(\frac{\partial h_i}{\partial y_j}\right)$, $(i, j = 1,2, \cdots, n)$.
Where $(y_1, y_2 \cdots, y_n)$, $(z_1, z_2 \cdots, z_n)$, $(\psi_1, \psi_2 \cdots, \psi_n)$, $(\Psi_1, \Psi_2 \cdots, \Psi_n)$ and ,$(h_1, h_2 \cdots, h_n)$are the components of Y, Z, ψ, Ψ, , and H respectively.

The instability of solutions for the vector differential equation with variable delay of fourth order has not been discussed by any authors to the best of our knowledge and this paper is considered the first to discussed this topic.
The symbol $\langle X, Y\rangle$ is used to denote the usual scalar product for any pair X, Y in $\mathbb{R}^n$, that is, $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$; thus $\langle X, X\rangle = \|X\|^2$, and $\lambda_i(A), i = 1,2 \cdots, n$, are the eigenvalues of the real symmetric $n \times n$ matrix A.

1- Preliminaries:

In order to introduce the main theorem we used the followings lemma

Lemma (2.1): (Bellman 1997) [1]

Let A be a real symmetric $n \times n$ matrix and

$$a' \geq \lambda_i(A) \geq a > 0, \qquad i = 1,2 \cdots, n$$

where a' and a are constants.
Then

$$a'\langle X, X\rangle \geq \langle AX, X\rangle \geq a\langle X, X\rangle,$$
$$a'^2\langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2\langle X, X\rangle.$$

Remark:

To prove the instability of the zero solution we use Lyapunov function $V = V(X, Y, Z, W)$ which have to satisfy the following three krasovskii properties [6]: (K1) In every neighborhood of (0, 0, 0, 0), there exists a point (ζ, ξ, η, μ) such that $V(\zeta, \xi, \eta, \mu) > 0$; (K2) the time derivative $\dot{V}$ along solution paths of systems (3),(4) are positive semi-definite; and (K3) the only solutions of the systems (3), (4) which satisfies $\dot{V}(t) = 0, t \geq 0$ is the trivial solution (0, 0, 0, 0).

Remark:

In order to guarantee the existence of the solution we assume that:
Consider the general autonomous delay differential system with finite delay:

$$\dot{x} = F(x_t), x_t = x(t + \theta) \qquad -r \leq \theta \leq 0, t \geq 0$$

Where $r \geq 0$, $F: G \to \mathbb{R}^n$ is a continuous and maps closed and bounded sets into bounded sets where G is an open subset of $C = C([-r, 0], \mathbb{R}^n)$ with $\|\phi\| = \max_{-r\leq s\leq 0}|\phi(s)|$, $\phi \in C$. If $x: [-r, A) \to \mathbb{R}^n$ is continuous, $0 < A \leq \infty$, then for each t in $[0, A)$, x_t in C is defined by:

$$x_t(s) = x(t + s) \qquad -r \leq s \leq 0, t \geq 0$$

It follows from these conditions on F that each initial value problem:

$$\dot{x} = F(x_t), \quad x_o = \phi \in G,$$

has a unique solution defined on some interval $[0, A), 0 < A \leq \infty$. This solution will be denoted by $x(\phi)(.)$ so that $x_o(\phi) = \phi$.

2- Main Result

The main result of this paper is the following two theorems:

Theorem (3.1):

Beside the basic assumptions of the Φ, Ω, ψ and Θ that appear in equation (1) we assume that there exist a constant a, a_1, a_2 and b such that following the conditions are hold:

$$a < \lambda_i\left(J_\Theta\left(Y(s)\right)\right) < b, \qquad \lambda_i(\Omega(X, Y, Z, W) < -a_1, \qquad \lambda_i\left(J_\psi(Z)\right) < -a_2$$

If

$$\sup_{0\leq t\leq\infty} \tau(t) \leq \frac{a_2 + a_1}{b}$$

Then the zero solution of equation (1) is unstable.

Proof:
We define the Lyapunov function $V = V(X, Y, Z, W) = V(X(t), Y(t), Z(t), W(t))$ as follows

$$V = \langle W, Z\rangle + \int_0^1 \langle\delta\Phi(\delta Z)Z, Z\rangle d\delta + \int_0^1 \langle\Theta(\delta Y), Y\rangle d\delta - \lambda \int_{-\tau(t)}^{0} \int_{t+s}^{t} \|Z(\theta)\|^2 \, d\theta \, ds$$

Where λ is a positive constant show that in the end of the proof, and $\int_{-\tau(t)}^{0} \int_{t+s}^{t} \|Z(\theta)\|^2 \, d\theta \, ds$ is non negative where s is a real variable. Hence the hypothesis of theorem (3.1) and $\Theta(Y) = \int_0^1 J_\Theta(\delta Y)Y d\delta$

$$\int_0^1 \langle\Theta(\delta Y), Y\rangle d\delta = \int_0^1 \int_0^1 \langle\delta_1 J_\Theta(\delta_1\delta_2 Y)Y, Y\rangle \, d\delta_2 d\,\delta_1 \geq \int_0^1 \int_0^1 \langle\delta_1 aY, Y\rangle d\delta_2 d\delta_1 = \frac{a}{2}\|Y\|^2$$

$$V(0, \in, 0,0) \geq \frac{a}{2}\|\in\|^2 > 0$$

The property (K1) of krasovskii property is verifies for all sufficiently small $\in$
Now derive V with respect to time and using system (3):

$$\dot{V} = -\langle \Phi(Z)W, \mathrm{Z}\rangle - \langle \Omega(\mathrm{X},\mathrm{Y},\mathrm{Z},\mathrm{W})\mathrm{Z},\mathrm{Z}\rangle - \langle \psi(Z),\mathrm{Z}\rangle - \langle \Theta(Y),\mathrm{Z}\rangle + \langle \int_{t-\tau(t)}^{t} J_{\Theta}\big(Y(s)\big)Z(s)ds\,,\mathrm{Z}\rangle + \langle \Phi(Z)W,\mathrm{Z}\rangle + \langle \Theta(Y),\mathrm{Z}\rangle$$

$$+\langle W,\mathrm{W}\rangle - \lambda\|Z\|^2\tau(\mathrm{t}) + (1-\acute{\tau}(\mathrm{t}))\lambda \int_{t-\tau(\mathrm{t})}^{t} \|Z(s)\|^2\ ds$$

Since

$$\frac{d}{dt}\int_0^1 \langle \Theta(\delta Y),\mathrm{Y}\rangle d\delta = \int_0^1 \delta\langle J_{\Theta}(\delta Y)Z,\mathrm{Y}\rangle d\delta + \int_0^1 \langle \Theta(\delta Y),\mathrm{Z}\rangle d\delta$$

$$= \int_0^1 \delta\frac{\partial}{\partial\delta}\langle \Theta(\delta Y),\mathrm{Z}\rangle d\delta + \int_0^1 \langle \Theta(\delta Y),\mathrm{Z}\rangle d\delta$$

$$= \delta\langle \Theta(\delta \mathrm{Y}),\mathrm{Z}\rangle\big|_0^1 = \langle \Theta(\mathrm{Y}),\mathrm{Z}\rangle$$

And

$$\frac{d}{dt}\int_0^1 \langle \delta\Phi(\delta Z)Z,\mathrm{Z}\rangle d\delta = \int_0^1 \langle \delta\Phi(\delta Z)W,\mathrm{Z}\rangle d\delta + \int_0^1 \delta^2\langle J_{\Phi}(\delta Z)ZW,\mathrm{Z}\rangle d\delta + \int_0^1 \delta\langle \Phi(\delta Z)Z,\mathrm{W}\rangle d\delta$$

$$= \int_0^1 \langle \delta\Phi(\delta Z)W,\mathrm{Z}\rangle d\delta + \int_0^1 \delta\frac{\partial}{\partial\delta}\langle \delta\Phi(\delta Z)W,\mathrm{Z}\rangle d\delta$$

$$= \delta^2\langle \Phi(\delta\mathrm{Z})\mathrm{W},\mathrm{Z}\rangle\Big|_0^1 = \langle \Phi(\mathrm{Z})\mathrm{W},\mathrm{Z}\rangle$$

And

$$\langle \int_{t-\tau(\mathrm{t})}^{t} J_{\Theta}\big(Y(s)\big)Z(s)ds\,,\mathrm{Z}\rangle \geq -\|Z\|\left\|\int_{t-\tau(\mathrm{t})}^{t} J_{\Theta}\big(Y(s)\big)Z(s)ds\right\|$$

$$\geq -\|Z\|b\left\|\int_{t-\tau(\mathrm{t})}^{t} Z(s)\right\|ds$$

$$\geq -\|Z\|b\int_{t-\tau(\mathrm{t})}^{t} \|Z(s)\|\,ds$$

$$\geq -\frac{b}{2}\tau(\mathrm{t})\|Z\|^2 - \frac{b}{2}\int_{t-\tau(\mathrm{t})}^{t} \|Z(s)\|^2\,ds$$

And

$$\frac{d}{dt}\int_{-\tau(\mathrm{t})}^{0}\int_{t+s}^{t} \|Z(\theta)\|^2\,d\theta\,ds = \|Z\|^2\tau(\mathrm{t}) - (1-\acute{\tau}(\mathrm{t}))\int_{t-\tau(\mathrm{t})}^{t} \|Z(s)\|^2\,ds$$

Now

$$\dot{V} \geq -\langle \Omega(\mathrm{X},\mathrm{Y},\mathrm{Z},\mathrm{W})\mathrm{Z},\mathrm{Z}\rangle - \frac{b}{2}\tau(\mathrm{t})\|Z\|^2 - \frac{b}{2}\int_{t-\tau(\mathrm{t})}^{t} \|Z(s)\|^2\,ds - \lambda\tau(\mathrm{t})\|Z\|^2 + (1-\acute{\tau}(\mathrm{t}))\lambda\int_{t-\tau(\mathrm{t})}^{t} \|Z(s)\|^2\ ds + \langle W,\mathrm{W}\rangle - \langle \psi(Z),\mathrm{Z}\rangle$$

$$\dot{V} \geq \langle W,\mathrm{W}\rangle + a_1\langle \mathrm{Z},\mathrm{Z}\rangle + a_2\langle \mathrm{Z},\mathrm{Z}\rangle - \frac{b}{2}\tau(\mathrm{t})\|Z\|^2 + \left((1-\acute{\tau}(\mathrm{t}))\lambda - \frac{b}{2}\right)\int_{t-\tau(\mathrm{t})}^{t} \|Z(s)\|^2\,ds - \lambda\tau(\mathrm{t})\|Z\|^2$$

$$\dot{V} \geq \langle W,W\rangle + \left(a_2 + a_1 - \left(\frac{b}{2}+\lambda\right)\tau(t)\right)\|Z\|^2 + \left(\big(1-\acute{\tau}(t)\big)\lambda - \frac{b}{2}\right)\int_{t-\tau(t)}^{t} \|Z(s)\|^2\,ds$$

Let $\left(1 - \acute{\tau}(\mathrm{t})\right) \geq \frac{1}{2}$, and $\lambda = \frac{b}{2}$
Then

$$\dot{V} \geq \left(a_2 + a_1 - \left(\frac{b}{2} + \frac{b}{2}\right)\tau(\mathrm{t})\right)\|Z\|^2$$
$$\dot{V} \geq \left(a_2 + a_1 - b\tau(\mathrm{t})\right)\|Z\|^2$$

If

$$\sup_{0 \leq t \leq \infty} \tau(\mathrm{t}) \leq \frac{\mathrm{a}_2 + \mathrm{a}_1}{\mathrm{b}}$$

then $\dot{V} \geq 0$ which satisfies the property (K2) of Krasovskii, on the other hand, $\dot{V} = 0$ if and only if $Z = 0 = Y = X = W$ for all $t \geq 0$, thus the property (K3) of Krasovskii is satisfies, so we conclude that the zero solution of equation (1) is unstable.
Now the second main result is:

Theorem (3.2):

Beside the basic assumptions of the $\mathrm{F}, \mathrm{G}, \Psi$ and H that appear in equation (2) we assume that there exist a constant $\mathrm{a}_1, \mathrm{b}_1, \mathrm{b}, \delta$ and a such that:

$$\mathrm{a} < \lambda_{\mathrm{i}}\left(\mathrm{J}_{\mathrm{H}}\left(\mathrm{Y(s)}\right)\right) < \mathrm{a}_1, \qquad \lambda_{\mathrm{i}}(\mathrm{A}) > b, \qquad \lambda_{\mathrm{i}}\left(\mathrm{J}_{\Psi}(\mathrm{Z})\right) < -\mathrm{b}_1$$

And $\frac{1}{4}\lambda_{\mathrm{i}}\left(\mathrm{F}\left(\mathrm{X(t)}, \mathrm{X}\left(\mathrm{t} - \tau(\mathrm{t})\right), \cdots, \mathrm{W(t)}, \mathrm{W}\left(\mathrm{t} - \tau(\mathrm{t})\right)\right)\right)^2 + \lambda_{\mathrm{i}}\left(\mathrm{G}\left(\mathrm{X(t)}, \mathrm{X}\left(\mathrm{t} - \tau(\mathrm{t})\right), \cdots, \mathrm{W(t)}, \mathrm{W}\left(\mathrm{t} - \tau(\mathrm{t})\right)\right)\right) < -\delta$
If

$$\sup_{0 \leq t \leq \infty} \tau(\mathrm{t}) \leq \frac{\delta + \mathrm{b}_1}{\mathrm{a}_1}$$

then the zero solution (2) is unstable.

Proof:

We consider the Lyapunov function
$\mathrm{V}_1 = \mathrm{V}_1(\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{W}) = \mathrm{V}_1(\mathrm{X(t)}, \mathrm{Y(t)}, \mathrm{Z(t)}, \mathrm{W(t)})$ as follows

$$\mathrm{V}_1 = \langle \mathrm{W}, \mathrm{Z} \rangle + \langle \mathrm{AX}, \mathrm{Y} \rangle - \lambda \int_{-\tau(\mathrm{t})}^{0} \int_{\mathrm{t+s}}^{\mathrm{t}} \|\mathrm{Z}(\theta)\|^2 \, \mathrm{d}\theta \, \mathrm{ds} + \int_0^1 \langle \mathrm{H}(\delta \mathrm{Y}), \mathrm{Y} \rangle \mathrm{d}\delta$$

Where λ is a positive constant which will be determined in the end of the proof, and s is a real variable such that $\int_{-\tau(\mathrm{t})}^{0} \int_{\mathrm{t+s}}^{\mathrm{t}} \|\mathrm{Z}(\theta)\|^2 \, \mathrm{d}\theta \, \mathrm{ds}$ is non negative.
Hence the hypothesis of theorem (3.2) and $\mathrm{H(Y)} = \int_0^1 \mathrm{J}_{\mathrm{H}}(\delta \mathrm{Y})\mathrm{Y}\mathrm{d}\delta$

$$\int_0^1 \langle \mathrm{H}(\delta \mathrm{Y}), \mathrm{Y} \rangle \mathrm{d}\delta = \int_0^1 \int_0^1 \langle \delta_1 \mathrm{J}_{\mathrm{H}}(\delta_1 \delta_2 \mathrm{Y})\mathrm{Y}, \mathrm{Y} \rangle \mathrm{d}\delta_2 \mathrm{d}\delta_1 \geq \int_0^1 \int_0^1 \langle \delta_1 \mathrm{aY}, \mathrm{Y} \rangle \mathrm{d}\delta_2 \mathrm{d}\delta_1 = \frac{\mathrm{a}}{2}\|\mathrm{Y}\|^2$$

$$\mathrm{V}(0, \in, 0, 0) \geq \frac{\mathrm{a}}{2}\|\in\|^2 > 0$$

The property (K1) of krasovskii property is verifies for all sufficiently small $\in$
Now derive V with respect to time and using system (4) we get:

$$\begin{aligned}\dot{V}_1 = {} & \langle W,W\rangle - \langle F\left(X(t),X(t-\tau(t)),\cdots,W(t),W(t-\tau(t))\right)W,Z\rangle - \langle G\left(X(t),X(t-\tau(t)),\cdots,W(t),W(t-\tau(t))\right)Z,Z\rangle \\ & - \langle H(Y),Z\rangle - \langle AX,Z\rangle - \langle \Psi(Z),Z\rangle + \langle AY,Y\rangle + \langle AX,Z\rangle - \lambda\tau(t)\|Z\|^2 + \left(1-\acute{\tau}(t)\right)\lambda \int_{t-\tau(t)}^{t} \left\|Z\left((\theta)\right)\right\|^2 \, d\theta \\ & + \langle H(Y),Z\rangle + \langle \int_{t-\tau(t)}^{t} J_H\left(Y(s)\right)Z(s)ds\,,Z\rangle\end{aligned}$$

By using that

$$\begin{aligned}\langle \int_{t-\tau(t)}^{t} J_H\left(Y(s)\right)Z(s)ds\,,Z\rangle & \geq -\|Z\| \left\| \int_{t-\tau(t)}^{t} J_H\left(Y(s)\right)Z(s)ds \right\| \\ & \geq -\|Z\|a_1 \left\| \int_{t-\tau(t)}^{t} Z(s) \right\| ds \geq -\|Z\|a_1 \int_{t-\tau(t)}^{t} \|Z(s)\|ds \\ & \geq -\frac{a_1}{2}\tau(t)\|Z\|^2 - \frac{a_1}{2}\int_{t-\tau(t)}^{t} \|Z(s)\|^2 ds\end{aligned}$$

Now

$$\begin{aligned}\dot{V}_1 \geq {} & \langle W,W\rangle - \langle F\left(X(t),X(t-\tau(t)),\cdots,W(t),W(t-\tau(t))\right)W,Z\rangle - \langle G\left(X(t),X(t-\tau(t)),\cdots,W(t),W(t-\tau(t))\right)Z,Z\rangle \\ & - \langle \Psi(Z),Z\rangle + \langle AY,Y\rangle - \frac{a_1}{2}\tau(t)\|Z\|^2 - \frac{a_1}{2}\int_{t-\tau(t)}^{t} \|Z(s)\|^2 ds - \lambda\tau(t)\|Z\|^2 \\ & + \left(1-\acute{\tau}(t)\right)\lambda \int_{t-\tau(t)}^{t} \left\|Z\left((\theta)\right)\right\|^2 \, d\theta\end{aligned}$$

Let $\left(1-\acute{\tau}(t)\right) \geq \frac{1}{2}$, and $\lambda = \frac{1}{2}a_1$, using the hypotheses of theorem (3.2)

Then

$$\begin{aligned}\dot{V}_1 \geq {} & \left\| W - \frac{1}{2}F\left(X(t),X(t-\tau(t)),\cdots,W(t),W(t-\tau(t))\right)Z \right\|^2 \\ & - \frac{1}{4}\langle F\left(X(t),X(t-\tau(t)),\cdots,W(t),W(t-\tau(t))\right)Z, F\left(X(t),X(t-\tau(t)),\cdots,W(t),W(t-\tau(t))\right)Z\rangle \\ & - \langle G\left(X(t),X(t-\tau(t)),\cdots,W(t),W(t-\tau(t))\right)Z,Z\rangle - \frac{a_1}{2}\tau(t)\|Z\|^2 + b\langle Y,Y\rangle + b_1\langle Z,Z\rangle - \frac{a_1}{2}\tau(t)\|Z\|^2\end{aligned}$$

$$\dot{V}_1 \geq (\delta + b_1)\|Z\|^2 - a_1\tau(t)\|Z\|^2$$

If

$$\sup_{0\leq t\leq\infty} \tau(t) \leq \frac{\delta + b_1}{a_1}$$

then $\dot{V}_1 \geq 0$ which satisfies the property (K2) of Krasovskii, on the other hand, $\dot{V} = 0$ if and only if $X = Z = 0 = Y = W$ for all $t \geq 0$, thus the property (K3) of Krasovskii is satisfies, we conclude that the zero solution of equation (1) is unstable.

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