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## RESEARCH ARTICLE

## Integrability under an Inner Boundary Complex Domain

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**Abstract**

The purpose of this paper is generating a sharp condition on boundary  $\Omega \in \mathbb{C}$  and  $w$  be a positive measurable function defined on  $\Omega$  which satisfy the assorted deterioration estimate at inner boundary of  $\Omega$  so the main result is Theorem (1), which comes as a corollary of the Koebe one quarter theorem and Koebe distortion theorem with discusses existence of positive constant  $K$  depending only on the modulus of  $z$  in  $\Omega$ , such that  $(1 - \phi z) \leq K|z|$ , for some  $z$  in  $\Omega$ , as well as, has been used this result as an essential tool to prove Theorems (2) and (3).

## INTRODUCTION

The univalent functions theory is old and quite wide topic in a complex analysis. As is known that the oldest complex tools in this aspect are: Koebe one quarter theorem and koebe distortion theorem hence shall focus on the role of each them in an examine the inner boundary of  $\Omega$  (be a bounded simply domain) through finding a constant  $K > 0$ , which considers as a special tool for restriction the diameter of given domain.

This work is related closely with similar lemma<sup>1</sup> which is quite known in polynomial approximation theory and has an important role to find conditions on  $\Omega \in \mathbb{C}$  (and  $w > 0$  be a measurable function defined on  $\Omega$  which led to  $H^p(\Omega, dA) = L_a^p(\Omega, dA)^2$ , for each  $p, 1 \leq p < \infty$ , we refer to [9], [10] and [11] for further more details.

**Definition (1) (Harmonic measure) [8]**

Let  $\Omega$  be a bounded, open domain in  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ ,  $n \geq 2$ , and let  $\partial\Omega$  denote the boundary of  $\Omega$ . Any finite real-valued continuous function  $f$  on  $\partial\Omega$ ,  $f: \partial\Omega \rightarrow \mathbb{R}$  corresponds to a unique function  $u(x)$  on the closure  $\bar{\Omega}$  of the region, is called a solution of the Dirichlet problem, if

- i.  $u$  is a continuous on  $\bar{\Omega}$ .
- ii.  $u$  is a harmonic in  $\Omega$ , that is  $\Delta u \equiv 0$  in  $\Omega$ .
- iii.  $u|_{\partial\Omega} = f$

A solution of the Dirichlet problem  $u$  corresponding to the continuous boundary function  $f$ , is called a harmonic extension of  $f$ , let us call it,  $u|_f = u(f)$ .

<sup>1</sup>For interested reader, we refer to [11] as a more convenient reference

<sup>2</sup> $H^p(\Omega, dA)$ - space consists of all the functions, that can be approximated arbitrarily in the  $L^p(\Omega, dA)$ -norm by a sequence of polynomials;  $L_a^p(\Omega, dA)$  denotes the class of all the functions in  $L^p(\Omega, dA)$  which are holomorphic in  $\Omega$ .

If the point  $x \in \Omega$  is assumed to be fixed, then by Riesz representation theorem<sup>3</sup>  $u(f)$  defined on the compact set  $C_c(\Omega)$  there exists a unique Borel measure  $\mu(x)$  at the point  $x$  on  $\Omega$ , define  $u_\mu \in C_c(\Omega)^*$  by

$$u_\mu(f) = \int f d\mu(x, \Omega),$$

for all  $f$  in  $C_c(\Omega)$ , and the measure  $\mu(x, \Omega)$  is called the Harmonic measure.

In particular, if the solution of the Dirichlet problem  $u$  corresponding to a Borel measurable  $E \subseteq \partial\Omega$  with the boundary data

$$f = \chi_E = \begin{cases} 1 & \text{if } \xi \in E, \\ 0 & \text{if } \xi \notin E. \end{cases}$$

such that  $f = 1_E$  that is,  $f$  takes the value 1 on the part  $E$  of  $\partial\Omega$  and  $f = 0$  on the remaining  $\partial\Omega$  such that, this solution is called the harmonic measure of  $E$ , and is denoted by  $\mu(x, E; \Omega)$ .

For much extra information about harmonic measure, we refer to [1], [6] and [15].

**Theorem (Koebe One-Quarter theorem) [5]**

The range of every function of class  $S^4$  contains the disk  $\{w: |w| < \frac{1}{4}\}$ .

**Lemma (1) [3, 9]**

Let  $E$  be a compact subset of the plane having connected complement and let  $g \in L(E, dA)$ , for some  $q > 1$ . If  $g^\wedge = 0$  identically in  $\mathbb{C} \setminus E$  then  $g^\wedge(z_o) = 0$  at every point  $z_o \in \partial E$  where

$$\int_E \frac{|g(z)|^q}{|z - z_o|} dA < \infty.$$

**Lemma (2) [2,3]**

Let  $E$  be a compact set with connected complement and let  $g \in L(E, dA)$ , for some  $1 < q \leq 2$ . If  $g^\wedge = 0$  identically in  $\mathbb{C} \setminus E$  and  $\xi$  is a point of  $E^o$  (the interior of  $E$ ) at a distance  $\delta(z) = \frac{1}{e}$  from  $\partial E$  then

$$|g^\wedge(\xi)| \leq C \left\{ g^*(\xi_o) \delta \log\left(\frac{1}{\delta}\right) + \left[ \Gamma_q(\delta) \int_{|z-\xi_o| \leq 4\delta} |g(z)|^q dA \right]^{\frac{1}{q}} \right\}$$

where  $g^*(\xi_o) = \sup_r (\pi r^2)^{-1} \int_{|z-\xi_o| < r} |g(z)| dA$  is the Hardy- Littlewood maximal function,  $\Gamma_q(\delta)$  is equal to  $\log \frac{1}{\delta}$  according to whether  $q = 2$  or  $q < 2$  and,  $C$  is a constant depending only on  $q$  and the diameter of  $E$ .

**Lemma (3)<sup>5</sup>**

There exist positive constants  $K_1$  and  $K_2$ , depending only on  $\delta(\phi^{-1}(0))$ , such that

$$K_1 \frac{1 - |\phi(z)|}{\delta(z)} \leq |\phi^{-1}(z)| \leq K_2 \frac{1 - |\phi(z)|}{\delta(z)},$$

where  $\phi^{-1}(0) = z$ , and  $\phi$  maps a simply connected domain  $\Omega$  conformally onto unit disk  $D$

Now, our starting point in this research paper with Theorem (2) and then we indicate how this theorem can be used to produce other results.

**Main results**

**Theorem (1)**

Let  $\phi$  be a conformal mapping of a simply connected domain  $\Omega$  onto the unit disk  $D$  then there is a constant  $K$  depending only on the modulus of  $z$  in  $\Omega$  such that

$$1 - |\phi(z)| \leq K\sqrt{|z|}, \text{ for some } z \in \Omega.$$

**Proof**

Let  $\Omega$  be a bounded simply connected domain, and  $\phi$  be a conformal map defined as follows:

$$\phi: \Omega \rightarrow D (|w| < 1).$$

So, the inverse function

$$\psi = \phi^{-1}: D \rightarrow \Omega.$$

Apply Distortion theorem Koebe to the inverse function  $\psi = \phi^{-1}(w)$ , to obtain that

$$\phi^{-1}(w) \in S, \text{ normalized by the conditions } \phi^{-1}(0) = 0, \text{ and } (\phi^{-1})'(0) = 1.$$

Then, for  $w \in D$ ;

<sup>3</sup> A straight-forward proof of Riesz representation theorem (cf. [16])

<sup>4</sup> Class  $S$  of functions  $\phi$  holomorphic and univalent in the unit disk  $D = \{z: |z| < 1\}$ , normalized by the conditions  $\phi(0) = 0$  and  $\phi'(0) = 1$ .

<sup>5</sup> A proof of Lemma can be found in ([14] pp.21-22).

$$\frac{|w|}{(1 + |w|)^2} \leq |\phi^{-1}(w)| \leq \frac{|w|}{(1 - |w|)^2}.$$

Fix  $w_o \in D$ , then

$$|\phi^{-1}(w_o)| \leq \frac{|w_o|}{(1 - |w_o|)^2} \dots \dots \dots (1)$$

Fix  $z_o \in \Omega$ . Then apply Koebe one-quarter theorem in order to show that the range of the function  $\phi^{-1}(w) \in S$  contains the disk  $\{\phi^{-1}(w): |\phi^{-1}(w)| < \frac{1}{4}\} \subseteq \Omega^o$  such that  $z_o \notin \{\phi^{-1}(w): |\phi^{-1}(w)| < \frac{1}{4}\}$ , which implies

$$|z_o| \geq \frac{1}{4} \quad \text{in } \Omega \dots \dots \dots (2)$$

Equation (2) can therefore be written as follows:

$$|z_o| \geq \frac{1}{4} |(\phi^{-1})'(0)| \quad \text{in } \Omega \text{ by condition } (\phi^{-1})'(0) = 1.$$

We can assume that  $w_o \in D$  is the image of  $-1$ , that is ;  $\phi(-1) = w_o$ .

By taking the inverse of both sides , we obtain

$$|\phi^{-1}(w_o)| = 1 \dots \dots \dots (3)$$

Distortion theorem which represents by equation (1) will now be applied to obtain

$$1 = |\phi^{-1}(w_o)| = \frac{|w_o|}{(1 - |w_o|)^2} = \frac{|w_o| |(\phi^{-1})'(0)|}{(1 - |w_o|)^2}$$

$$1 \leq \frac{|w_o| |(\phi^{-1})'(0)|}{(1 - |w_o|)^2} \leq \frac{4|z_o||w_o|}{(1 - |w_o|)^2} \quad \text{by equation (2)}$$

Since,  $z_o$  and  $w_o$  are arbitrary points, this implies that

$$1 \leq \frac{4|z||w|}{(1 - |w|)^2} < \frac{C|z|}{(1 - |w|)^2} = \frac{C|z|}{(1 - |\phi(z)|)^2}$$

Hence,

$$(1 - |\phi(z)|)^2 \leq C|z|$$

Finally, we obtain

$$1 - |\phi(z)| \leq K\sqrt{|z|} \blacksquare$$

**Theorem (2)**

Let  $\Omega$  be a bounded simply connected domain, whose boundary is a class  $C^1$  Jordan curve. If  $\phi$  is a conformal map of  $\Omega$  to the unit disk  $D(|w|<1)$ , then

$$\iint_{\Omega} |\phi'|^p \, dx \, dy < \infty, \text{ for all } p < 2.$$

**Proof**

We shall assume that  $z_o \in \Omega$ , and  $\phi(z_o) = 0$ . It can then be inferred from the co-area formula<sup>6</sup>.

Let  $\phi$  be a conformal mapping of  $\Omega$  (simply connected domain) onto the open unit disk  $D(z:|z|<1)$ , that is;  $\phi: \Omega \rightarrow D$  with  $\mathcal{M}$  be a measurable subset of  $D$ .

Let  $g(z): B \subseteq \Omega \rightarrow [0, \infty)$ , where  $B = \{z \in \Omega: |\phi(z)| = r\}$  be a measurable function defined on the measurable set  $B$  in  $\Omega$ , as the following

$$g(z) = \frac{\chi_{B(z)}}{|\phi'(z)|}: \Omega \rightarrow [0, \infty)$$

where  $\chi_{B(z)}: B \subseteq \Omega \rightarrow \{0,1\}$  is the characteristic function.

$$\chi_{B(z)} = \begin{cases} 1 & \text{if } z \in B, \\ 0 & \text{if } z \notin B. \end{cases}$$

Because we have to calculate the integral over  $B$  and we know that  $0 < r \leq 1$  so,  $\chi_B = 1, z \in B$  such that

$$\begin{aligned} \iint_{\Omega} \frac{\chi_{B(z)}}{|\phi'(z)|} |\phi'| \, dx \, dy &= \int_0^1 \left( \int_{B=\{z \in \Omega: |\phi(z)|=r\}} \frac{\chi_{B(z)}}{|\phi'(z)|} r \, d\theta \right) dr \\ \iint_{\Omega} dx \, dy &= \int_0^1 \left( \int_{B=\{z \in \Omega: |\phi(z)|=r\}} \frac{1}{|\phi'(z)|} r \, ds \right) dr. \end{aligned}$$

Now we have

<sup>6</sup> A straight-forward proof of this assertion is in [13].

$$\begin{aligned} \iint_{\Omega} |\phi'|^p dx dy &= \int_0^1 \left( \int_{B=\{z \in \Omega: |\phi(z)|=r\}} \frac{1}{|\phi'(z)|} |\phi'|^p ds \right) dr \\ &= \int_0^1 \left( \int_{B=\{z \in \Omega: |\phi(z)|=r\}} |\phi'|^{p-1} ds \right) dr \\ &= \int_0^1 |\phi'|^{p-1} \left( \int_0^{2\pi r} ds \right) dr \\ &= \int_0^1 2\pi r |\phi'|^{p-1} dr \end{aligned}$$

By Lemma (3) we have  $|\phi'| \leq C \frac{1-|\phi|}{\delta(z)}$ , so let us assume that  $\delta(z) = |z|$  such that

$$\begin{aligned} \iint_{\Omega} |\phi'|^p dx dy &= \int_0^1 2\pi r |\phi'|^{p-1} \\ &\leq \int_0^1 2\pi r \frac{(1-r)^{p-1}}{|z|^{p-1}} dr \end{aligned}$$

In theorem (1) we deduced that  $(1 - |\phi|) \leq K_1 \sqrt{|z|}$  which implies

$$\frac{K_2}{|z|^{p-1}} \leq \frac{1}{(1-r)^{2(p-1)}}.$$

Hence,

$$\begin{aligned} \int_0^1 2\pi r \frac{(1-r)^{p-1}}{|z|^{p-1}} dr &\leq \int_0^1 2\pi r \frac{(1-r)^{p-1}}{(1-r)^{2(p-1)}} dr \\ &= \int_0^1 2\pi r (1-r)^{-(p-1)} dr, \end{aligned}$$

when  $-(p-1) + 1 > 0 \Rightarrow p < 2$ . ■

**Theorem (3)**

Let  $E$  be a compact subset of the plane having connected complement,  $D \subseteq E$  be a connected domain, let  $g^\wedge$  be a Cauchy transform of a function  $g$ , where  $g \in L^q(E, dA)$  for some  $1 < q \leq 2$ , if  $g^\wedge(z)$  is an identically zero in  $\mathbb{C}E$  and there exist a non-decreasing sequence  $\partial D_i$  in  $D$  then

$$\int_{\partial D_i} \left( \frac{|g^\wedge(z)|}{\sqrt{|z|}} \right) d\mu_i \text{ exists and is finite on } \partial D_i$$

Let  $D$  be a connected domain  $D \subseteq E$ . Fix arbitrary point  $\xi \in D$  and assume  $\mu$  be the harmonic measure on  $\partial D$  representing  $\xi$ .

Draw  $D_i$  be a non-decreasing sequence such that  $\xi \in D_i$  for all  $i$  and  $\cup D_i$  fill up  $D$ .

Given  $g \in L^q(E)$ , then  $g^\wedge$  vanishes identically<sup>7</sup> in the complementary of  $E$  since the integral  $g$  with any polynomial in  $H^p(E, dA)$  is equal to zero, so  $|g^\wedge(\xi)|$  is bounded in  $E$  at Euclidean distance (see lemma (2)) that is;  $|g^\wedge(\xi)|$  is bounded in  $D$ .

Hence, multiply and divide  $|g^\wedge(z)|$  by  $\sqrt{\delta(z)}$  where  $\delta(z) = \text{dis}(z, \partial D_i) = |z|$  this yields the identity below:

$$\begin{aligned} \log \frac{|g^\wedge(z)|\sqrt{|z|}}{\sqrt{|z|}} &= \log \sqrt{|z|} + \log \frac{|g^\wedge(z)|}{\sqrt{|z|}} \\ \log |g^\wedge(z)| &= \log \sqrt{|z|} + \log \frac{|g^\wedge(z)|}{\sqrt{|z|}} \end{aligned}$$

Integrate the last quantity over  $\partial D_i$  with respect to harmonic measure  $d\mu_i$  this will imply,

$$\int_{\partial D_i} \log |g^\wedge(z)| d\mu_i = \int_{\partial D_i} \log \sqrt{|z|} d\mu_i + \int_{\partial D_i} \log \frac{|g^\wedge(z)|}{\sqrt{|z|}} d\mu_i$$

Here, we will pay particular attention to the second integral.

As known in (pp. 145 [3]) that,

$$\int_{\partial D_i} \log \frac{|g^\wedge(z)|}{\sqrt{|z|}} d\mu_i < \int_{\partial D_i} \frac{|g^\wedge(z)|}{\sqrt{|z|}} d\mu_i \dots \dots \dots (4)$$

It is known in measure theory<sup>8</sup>

<sup>7</sup> we refer the reader to see (pp.119 [2])

<sup>8</sup> For interested reader we refer to [7] and definition () in appendix B.

$$\begin{aligned} \sum_{j=1}^n \frac{d}{dx_j} \int_{\partial D_i} \frac{|g^\wedge(z)|}{\sqrt{|z|}} d\mu_i &= \int_{\partial D_i} \sum_{j=1}^n \frac{\partial}{\partial x_j} \left( \frac{|g^\wedge(z)|}{\sqrt{|z|}} \right) d\mu_i \\ &= \int_{\partial D_i} \sum_{j=1}^n \nabla \left( \frac{|g^\wedge(z)|}{\sqrt{|z|}} \right) d\mu_i \end{aligned}$$

Hence, we have

$$\int_{\partial D_i} \sum_{j=1}^n \nabla \left( \frac{|g^\wedge(z)|}{\sqrt{|z|}} \right) d\mu_i = \int_{\partial D_i} \sum_{j=1}^n |g^\wedge(z)| \nabla \left( \frac{1}{\sqrt{|z|}} \right) d\mu_i + \int_{\partial D_i} \sum_{j=1}^n \frac{1}{\sqrt{|z|}} \nabla (|g^\wedge(z)|) d\mu_i \dots (5)$$

This implies that

$$\left| \int_{\partial D_i} \sum_{j=1}^n \nabla \left( \frac{|g^\wedge(z)|}{\sqrt{|z|}} \right) d\mu_i \right| < \int_{\partial D_i} |g^\wedge(z)| \nabla \left( \frac{1}{\sqrt{|z|}} \right) |d\mu_i| + \int_{\partial D_i} \frac{1}{\sqrt{|z|}} \nabla (|g^\wedge(z)|) |d\mu_i| \dots (6)$$

Now we have to define a Green function which is a harmonic function on  $D_i$  and remove the disk  $|z - \xi| \leq r_0$  from  $D_i$  such that we get the disk  $|z - \xi| \leq r_0$  in  $D'_i$ ; has a smooth boundary will lead to generate harmonic function on  $D_i$  with singularity (pole) at  $\xi$ . In this case should be define the harmonic function  $d\mu_i = \frac{\partial G_i}{\partial n} |dz|$  and let  $\phi_i$  be a conformal map of  $D_i$  onto unit disk  $D = \{w : |w| < 1\}$  with  $\phi_i(\xi) = 0$ ; and as clear  $\psi_i = \phi_i^{-1}$ , which satisfies the following:

- a.  $|\nabla G_i| \leq C|\phi'_i|$  on  $D'_i$ .
- b.  $|\psi'_i| \geq C(1 - |w|)$ .

Applying Hölder inequality to estimate  $I_1$  as follows:

$$\begin{aligned} I_1 &= \int_{\partial D_i} \left| |g^\wedge(z)| \nabla \left( \frac{1}{\sqrt{|z|}} \right) \right| |d\mu_i| \\ &= \int_{\partial D'_i} |g^\wedge(z)| \nabla \left( \frac{1}{\sqrt{|z|}} \right) |\nabla G_i| dA \\ &\leq \left( \int_{\partial D'_i} |g^\wedge|^q dA \right)^{\frac{1}{q}} \left( \int_{\partial D'_i} \left| \nabla \left( \frac{1}{\sqrt{|z|}} \right) \nabla G_i \right|^p dA \right)^{\frac{1}{p}} \\ &= \|g^\wedge\| \left( \int_{\partial D'_i} \left| \nabla \left( \frac{1}{\sqrt{|z|}} \right) \nabla G_i \right|^p dA \right)^{\frac{1}{p}}, \end{aligned}$$

hence we obtain

$$\begin{aligned} I_1^p &\leq C_4 \int_{\partial D'_i} \left| \nabla \left( \frac{1}{\sqrt{|z|}} \right) \nabla G_i \right|^p dA \\ &= C_4 \int_{\partial D'_i} \left| \nabla \left( \frac{1}{\sqrt{|z|}} \right) \right|^p \|\nabla G_i\|^p dA \\ &\leq C_4 \int_{\partial D_i} \phi_i^{p-2} \left| \nabla \left( \frac{1}{1 - \phi'_i} \right) \right|^p |\phi'_i|^2 dA \\ &= C_4 \int_{\partial D_i} \frac{1}{|\psi'_i|^{p-2}} \left| \nabla \left( \frac{1}{1 - r} \right) \right|^p dA \\ &= C_4 \left[ \frac{(1 - r)^{-3p+3}}{-3p + 3} \right]_0^1 \end{aligned}$$

such that when  $-3p + 3 > 0$ , this implies to  $p < 1$ .

It is a consequence of Hölder inequality<sup>9</sup> and Calderón-Zygmund theorem on the continuity of singular integral operators, cf.[10] & (pp.564 [4]), that

<sup>9</sup> For further information about the role of Hölder inequality

$$\begin{aligned}
I_2 &= \int_{\partial D_i} \left| \frac{\nabla(|g^\wedge(z)|)}{\sqrt{|z|}} \right| |d\mu_i| \\
&= \int_{\partial D_i} |\nabla|g^\wedge(z)|| \left| \frac{|\nabla G_i|}{\sqrt{|z|}} \right| dA \\
&\leq C_1 \left( \int_{\partial D_i} \left| \frac{\nabla G_i}{\sqrt{|z|}} \right|^p dA \right)^{\frac{1}{p}} \\
&\leq C_1 \int_{D_i} \frac{|\phi_i'|^{p-2}}{1-|\phi_i|} |\phi_i'|^2 dA \quad \text{by inequality (a)} \\
&= \int_{|w|<1} \frac{1}{|\psi_i'|^{p-2}} \frac{1}{(1-|w|)} dA \quad \text{by inequality (b)} \\
&= C_3 \left[ \frac{(1-r)^{2-p}}{2-p} \right]_0^1
\end{aligned}$$

when  $2 - p > 0 \Rightarrow p < 2$ .

Finally, the quantity

$$\int_{\partial D_i} \left( \frac{|g^\wedge(z)|}{\sqrt{|z|}} \right) d\mu_i \text{ exists and is finite on } \partial D_i \blacksquare$$

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