ISSN 2320-5407 International Journal of Advanced Research (2015), Volume 3, Issue 12, 884 – 892

ISSN NO. 2320-5407

Journal homepage: http://www.journalijar.com

INTERNATIONAL JOURNAL OF ADVANCED RESEARCH

RESEARCH ARTICLE

Solving Fredholm Integral Equation of the Second Kind with A Single Core

Ali Mohebi , and Hamid Mesgarani

Faculty of Basic Sciences, Applied Mathematics Group, University of Shahid Rajaee, Tehran, Iran

Manuscript Info

Manuscript History:

Received: 12 October 2015
Final Accepted: 25 November 2015
Published Online: December 2015

Key words:

Integral equation, single core, Legendre polynomials, orthogonal unit.

***Corresponding Author**

Ali Mohebi

Abstract

Integral equation with a single core has a special importance in different fields of science. Hence, in this article we have tried to obtain an estimation of a single core of integral equation by using correct method of sectioning a network of points. To this end, for obtaining a solution for integral equation, orthogonal base unit generated based on Legendre polynomials is used and the number of operations are reduced due to orthogonality of the above bases and operational matrixes are appeared as diagonal. In the end, obtained numerical results indicate to a high level of accuracy for the estimation.

INTRODUCTION

The following Fredholm integral equation of the second kind with a single core is considered:

$$u(t) - \int_0^1 \frac{u(s)ds}{(t-s)^{\alpha}} = f(t), \quad 0 \le t \le 1, \qquad 0 < \alpha \le 1 \tag{1}$$

Where, $u(t)$ ، $f(t) \in l^2[0,1]$ is the solution to the unknown integral equation. Assuming that the solution to integral equation [1] is located in $l^2[0,1]$, for selecting a network points, horizontal axis in interval of $[0,1]$ is divided into t_m ،...،t_1 ،t_0 and also, vertical axis in interval of $[0,1]$ is divided into s_m ،...،s_1 ،s_0. On a square like I_{ij}, the core is estimated by using Legendre base. Off course, a multiple wavelet orthogonal base unit generated based on Legendre polynomials can be used as well, that the Legendre base is a special case of that. for further studies about the way of generating the above multiple wavelet refer to [1]. For instances of the application of Legendre polynomials and multiple wavelet generated based on Legendre polynomials for solving differential - integral equations and non-linear integral equations of Hammerstein refer to [5,7]. Equation (1) has been also solved by [4,6], however, here with a new approach, we will show the efficiency of our method. For this purpose, see the network of points in figure 1.

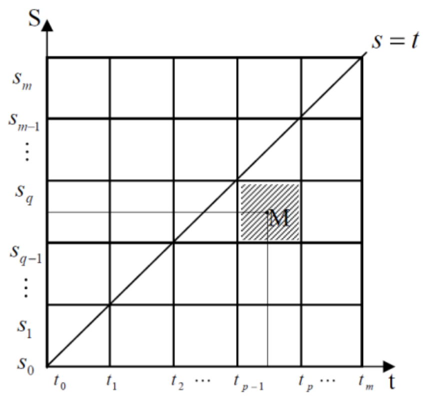

Figure 1.

By square I_{pq}, we refer to a square which is generated from multiplication of pth distance on t axis and qth distance on s axis. Therefore, we have the following:

$$[t_0,t_1]\times[S_0,S_1]=I_{11}$$
$$\vdots$$
$$[t_{p-1},t_p]\times[S_{q-1},S_q]=I_{pq}\,.$$

2. Core Estimation

We consider the dimensional space of X_n against n, and $k(t,s)$ core in the space of $X_n\times X_n$ is estimated as below:

$$k(t,s)=\sum_{i=1}^{n}\sum_{j=1}^{n}\kappa_{ij}b_i(t)b_j(s)$$

Orthogonal base unit of $\{b_i\}_{i=1}^n$ can be selected from Legendre polynomials orthogonal unit or multiple wavelet orthogonal unit generated based on Legendre polynomials. Hence, core estimation confidents are as per following:

$$\kappa_{ij}=\int_0^1\int_0^1 k(t,s)b_i(t)b_j(s)dtds=\int_0^1\int_0^1\left[\frac{b_i(t)b_j(s)}{(t-s)^{\alpha}}\right]dtds=\int_0^1\int_0^1 g_{ij}(t,s)dtds \tag{2}$$

For calculating integral (2) on s=t line, we are faced with the problem of having a single core. For solving this problem as per figure 1, with assuming $s_i=t_i$ for $i=0,1,\ldots,m$ and integrating the squares above and below s=t line and on squares for which s=t line crosses their diameter, integral (2) is calculated. Therefore, we have:

$$\kappa_{ij}=\int_0^1\int_0^1 g_{ij}(t,s)dtds=\sum_{\substack{p,q=1\\p\neq q}}^{m}\iint_{I_{pq}} g_{ij}(t,s)dtds+\sum_{p=1}^{m}\iint_{I_{pq}} g_{ij}(t,s)dtds\,. \tag{3}$$

2-1 Calculating integral equation on squares not located on s=t line

Points on s and t axes are assumed to have equal distances and therefore, we have:

$$t_p = t_0 + ph = ph$$
$$S_p = S_0 + ph = ph. \quad (4)$$

For calculating the integration of function $g_{ij}(t,s)$on square I_{pq}, we proceed accoridng to figure 2, in which M is the center of I_{pq} square ;

$$\iint_{I_{pq}} g_{ij}(t,s)dtds \approx (I_{pq} \quad \textbf{Square area} \quad)g_{ij}(t_M, S_M)$$

$$= (t_p - t_{p-1})(s_q - s_{q-1})g_{ij}(t_p - \frac{h}{2}, s_q - \frac{h}{2})$$

Given the fact that points have equal distance and considering equation (4), we have the following:

$$= h^2 g_{ij}\left(\frac{(2p-1)h}{2}, \frac{(2q-1)h}{2}\right)$$

And considering $h = \frac{1}{m}$, we have:

$$= \frac{1}{m^2} g_{ij}\left(\frac{(2p-1)}{2m}, \frac{(2q-1)}{2m}\right). \quad (5)$$

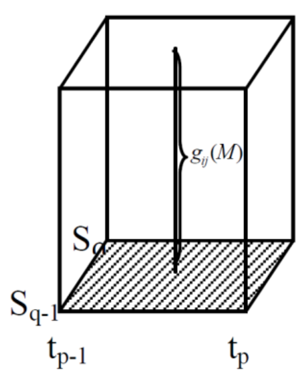

Figure 2.

2-2 Calculating integral on squares located on s = t line

Considering figure 3, square I_{pp} is divided into two triangles above and below $s=t$ line and every time the area of the triangle is multiplied into the value of function $g_{ij}(t,s)$ at the center of the triangle. For this purpose, coordinates of A, B, C, D are considered as below:

$$A=(t_{p-1},S_{p-1}),B=(t_p,S_{p-1}),C=(t_p,s_p),D=(t_{p-1},s_p)$$
$$,S_{ABC}=\frac{h^2}{2},S_{ADC}=\frac{h^2}{2}.$$

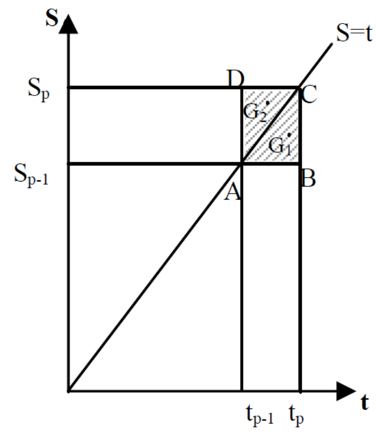

Figure 3.

Coordinates of G_1,G_2 in figure 3 are introduced as below:

$$G_1=\left(\frac{t_A+t_B+t_C}{3},\frac{s_A+s_B+s_C}{3}\right)=\left(\frac{t_{p-1}+t_p+t_p}{3},\frac{2s_{p-1}+s_p}{3}\right)$$
$$=\left(\frac{(p-1)h+2ph}{3},\frac{2(p-1)h+ph}{3}\right)$$
$$=\left(\frac{(3p-1)h}{3},\frac{(3p-2)h}{3}\right) \quad (6)$$

Similarly we have:

$$G_2=\left(\frac{(3p-2)h}{3},\frac{(3p-1)h}{3}\right). \quad (7)$$

Considering (6), (7) and $h=\frac{1}{m}$, we have:

$$\iint_{I_{pp}} g_{ij}(t,s)dtds = (S_{ABC})g_{ij}(G_1) + S_{ADC}\, g_{ij}(G_2) = \frac{1}{2m^2}\left[g_{ij}\left(\frac{(3p-1)}{3m}, \frac{(3p-2)}{3m}\right) + g_{ij}\left(\frac{(3p-2)}{3m}, \frac{(3p-1)}{3m}\right)\right]. \tag{8}$$

With considering equations (3), (5) and (8) , estimation coeffieicents of core, that is, κ_{ij}, are obtained as below:

$$\kappa_{ij} = \sum_{\substack{p,q=1\\ p\neq q}}^{m} \frac{1}{m^2} g_{ij}\left(\frac{(2p-1)}{2m}, \frac{(2q-1)}{2m}\right) + \sum_{p=1}^{m} \frac{1}{2m^2}\left[g_{ij}\left(\frac{(3p-1)}{3m}, \frac{(3p-2)}{3m}\right) + g_{ij}\left(\frac{(3p-2)}{3m}, \frac{(3p-1)}{3m}\right)\right]. \tag{9}$$

From (9) , coefficients of κ_{ij} for core estimation with m is found to be large enough, in which $p \neq q$ below and above s = t line is calculated as below:

$$\kappa_{ij} = \sum_{p=1}^{m-1}\sum_{q=p+1}^{m} \frac{1}{m^2} g_{ij}\left(\frac{(2p-1)}{2m}, \frac{(2q-1)}{2m}\right) + \sum_{q=1}^{m-1}\sum_{p=q+1}^{m} \frac{1}{m^2} g_{ij}\left(\frac{(2p-1)}{2m}, \frac{(2q-1)}{2m}\right) + \sum_{p=1}^{m} \frac{1}{2m^2}\left[g_{ij}\left(\frac{(3p-1)}{3m}, \frac{(3p-2)}{3m}\right) + g_{ij}\left(\frac{(3p-2)}{3m}, \frac{(3p-1)}{3m}\right)\right]. \tag{10}$$

Now, given the value of α at the core of $k(t,s) = \frac{1}{(t-s)^{\alpha}}$ and $g_{ij}(t,s)$, κ_{ij} can be calculated.

$$\kappa_{ij} = \sum_{p=1}^{m-1}\sum_{q=p+1}^{m} \frac{1}{m^2} \frac{b_i\left(\frac{2p-1}{2m}\right) b_j\left(\frac{2q-1}{2m}\right)}{\left(\frac{2p-1}{2m} - \frac{2q-1}{2m}\right)^{\alpha}} + \sum_{q=1}^{m-1}\sum_{p=q+1}^{m} \frac{1}{m^2} \frac{b_i\left(\frac{2p-1}{2m}\right) b_j\left(\frac{2q-1}{2m}\right)}{\left(\frac{2p-1}{2m} - \frac{2q-1}{2m}\right)^{\alpha}} + \sum_{p=1}^{m} \frac{1}{2m^2}\left[\frac{b_i\left(\frac{3p-1}{3m}\right) b_j\left(\frac{3p-2}{3m}\right)}{\left(\frac{3p-1}{3m} - \frac{3p-2}{3m}\right)^{\alpha}} + \frac{b_i\left(\frac{3p-2}{3m}\right) b_j\left(\frac{3p-1}{3m}\right)}{\left(\frac{3p-2}{3m} - \frac{3p-1}{3m}\right)^{\alpha}}\right].$$

With simplifying the above equations, κ_{ij} is introduced as below:

$$\kappa_{ij} = \sum_{p=1}^{m-1}\sum_{q=p+1}^{m} \frac{b_i\left(\frac{2p-1}{2m}\right) b_j\left(\frac{2q-1}{2m}\right)}{m^{(2-\alpha)}(p-q)^{\alpha}} + \sum_{q=1}^{m-1}\sum_{p=q+1}^{m} \frac{1}{m^2}\frac{b_i\left(\frac{2p-1}{2m}\right) b_j\left(\frac{2q-1}{2m}\right)}{m^{(2-\alpha)}(p-q)^{\alpha}}$$
$$+\sum_{p=1}^{m}\frac{1}{2m^{(2-\alpha)}}\left[3^{\alpha} b_i\left(\frac{3p-1}{3m}\right) b_j\left(\frac{3p-2}{3m}\right) + (-3)^{\alpha} b_i\left(\frac{3p-2}{3m}\right) b_j\left(\frac{3p-1}{3m}\right)\right] \tag{11}$$

In equation (11), multiple wavelet bases, in the case that the interval of $[0,1]$ is not divided into smaller intervals, are same as Legendre polynomials orthogonal unit on interval $[0,1]$ which are introduced in section 3, below.

3. Introducing Legendre orthogonal unit polynomials

Legendre orthogonal unit polynomials based on Rodrigues formula are presented as below. (for further information see [2,3])

$$L_n(t) = \frac{(-1)^n 2^n}{n!}\frac{d^n}{dt^n}\left[(1-t^2)^n\right], \quad t \in [-1,1],\ w(t) = 1. \tag{12}$$
$$L_0(t) = 1\ ,\ \ L_1(t) = t$$
$$(n+1)L_{n+1}(t) = (2n+1)t\,L_n(t) - nL_{n-1}(\mathrm{t}), \quad n \geq 1.$$

Also we have:

$$\int_{-1}^{1} L_m(t)L_n(t)\,dt = \begin{cases} 0 & m \neq n, \\ \dfrac{2}{2n+1} & m = n. \end{cases}$$

For Legendre orthogonal unit polynomials, we act as below:

$$\int_{-1}^{1} (\alpha L_m(t))\,(\alpha L_n(t))\,dt = \begin{cases} 0 & m \neq n, \\ 1 & m = n. \end{cases}$$

Therefore, we have:

$$\alpha = \sqrt{\frac{2n+1}{2}}\ ,\ L_0(t) = \sqrt{\frac{1}{2}}\ ,\ L_1(t) = \sqrt{\frac{3}{2}}\,t, \ldots .$$

3-1- Introducing Legendre orthogonal unit polynomials on $[0,1]$

With replacing the variable of $2t-1$ with t in (12), we have:

$$L_n(t)=\frac{(-1)^n 2^n}{n!}\frac{d^n}{dt^n}\left[(t-t^2)^n\right], \quad t\in[0,1]. \tag{13}$$

For generating Legendre orthogonal unit polynomials in equation (13), the condition of $\int_0^1(\alpha_j b_j(t))^2\,dt=1$ is used and for example, values of α_j are presented for $7 ,\ldots,1,0=j$ as below:

$$\alpha_0=1\ ,\ \alpha_1=\frac{\sqrt{3}}{2}\ ,\ \alpha_2=\frac{\sqrt{5}}{4}\ ,\ \alpha_3=\frac{\sqrt{7}}{8}\ ,\ \alpha_4=\frac{3}{16}\ ,\ \alpha_5=\frac{\sqrt{11}}{32}\ ,\ \alpha_6=\frac{\sqrt{13}}{64}\ ,\ \alpha_7=\frac{\sqrt{15}}{128}.$$

Also, $b_j(t)$ for $7 ,\ldots,1,0=j$ which is used in this article, is written as below:

$$\begin{aligned} &L_0(t)=1,\\ &L_1(t)=\sqrt{3}(2t-1),\\ &L_2(t)=\sqrt{5}(6t^2-6t+1),\\ &L_3(t)=\sqrt{7}(20t^3-30t^2+12t-1),\\ &L_4(t)=3(70t^4-140t^3+90t^2-20t+1). \end{aligned} \tag{14}$$

4. Numerical results

Numerical results are obtained by using *Mathematica ,5.1* software.

Example 4-1

The below problem of Fredholm integral equation of the second type with a single core is solved with the above method:

$$\begin{cases} u(t)-\int_0^1\frac{u(s)\,ds}{(t-s)^1}=1^2\left(1+Ln\left(\frac{1-t}{t}\right)\right)+t+\frac{1}{2}\\ u(t)=t^2 \qquad \text{: the real answer}\end{cases}$$

For solving example 4-1, Legendre base in the case of $n=3,4$ is used and the dimension of the solved equation is equal to 3 and 4, respectively and m is the number of estimation points on vertical and horizontal axes for core estimation. In table 1, it is seen that the obtained numerical solution has a high level of accuracy.

Table 1.

X_n	n	m	Error
S_0^3	3	300	8.2×10^{-6}
S_0^4	4	100	1.3×10^{-7}

Example 4-2

The following problem of Fredholm integral equation of the second type with a single core is solved with the above method:

$$\begin{cases} u(t)-\int_0^1 \frac{u(s)\,ds}{(t-s)^{1/3}} = t^2\left(1+\frac{3}{2}\left((t-1)^{2/3}-t^{2/3}\right)-\frac{6}{5}t\left((t-1)^{5/3}-t^{5/3}\right)t\,\frac{3}{8}\left((t-1)^{8/3}-t^{8/3}\right)\right) \\ u(t)=t^2 \quad \text{: the real answer} \end{cases}$$

For solving this problem, Legendre base of $n=3,4$ is used and the dimension of the solved equation is equal to 3 and 4, respectively and m is the number of estimation points on vertical and horizontal axes for core estimation. In table 2, it is seen that although the dimension of the equation is low, the obtained numerical solution has a high level of accuracy.

Table 2.

X_n	n	m	Error
S_0^3	3	20	9.0×10^{-5}
S_0^4	4	17	1.0×10^{-6}

Reference:

[1] B. K. Alpert, A class of Bases in L2 for the Sparse Representation of Integral Operators, SIAM J. Math. 24(1993), 246-262.

[2] K. Atkinson, An Introduction to Numerical Analysis University of Iowa, John Wiley & Sons, (1988).

[3] T. S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Reach Science Publishers, (1978).

[4] A. Gerasoulis, Piecwise-polynomial quadrarures for Cauchy singular Integrals, SIAM. J. Numer. Anal. 23(1986) 891-901.

[5]M. Maleknejad, M. Rabbani, N. Aghazadeh and M. Karami, A wavelet Petrov-Galerkin method for solving Integro-Differential Equation, International Jounrnal of Computer Mathematics. Vol. 00, No. 0, Month 2008, 1-19.

[6] F. A. Potra and E. Venturino, low-order methods for Cauchy principle value integrals with endpoint singularitices, IMAJ. Numer. Anal. 14(1994).

[7] M. Rabbani and K. Maleknejad, A Modification for Solving Fredholm-Hammerstein Integral

Equations Using Wavelet Basis, Kybernetes: The International Journal of Systems. Cybernetics and management. Vol. 38 Nos 3/4, 2009 pp. 615-620.