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RESEARCH ARTICLE

A PICARD-S ITERATION SCHEME FOR APPROXIMATING FIXED POINT OF ALMOST CONTRACTION MAPPINGS

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Abstract

In this paper we study the convergence analysis of a Picard-S iterative method for the almost contraction mappings. Furthermore, we discussed the uniqueness of the fixed point. Also, we compare the rates of convergence between Picard-S and modified SP iterations for the aforementioned class of mappings.

INTRODUCTION

Many mathematical problems, originating from various branches of mathematics, can be equivalently formulated as fixed point problems, meaning that one has to find fixed points of a certain operator, on the one hand, and the fact that contractive (Lipschitzian) type conditions naturally arise for many of these problems, on the other hand, the metrical fixed point theory has developed significantly in the second part of the 20th century, [3].

Convergence analysis of iterative methods has an important role in the study of iterative approximation of fixed point theory. Fixed point iteration methods may exhibit radically different behaviors for various classes of mappings. While a particular fixed point iteration method is convergent for an appropriate class of mappings, it may not be convergent for the others. Due to various reasons, it is important to determine whether an iteration method converges to fixed point of a mapping. In many cases, there can be two or more than two iteration procedures approximating to a fixed point of a mapping. In such cases, the critical and important point is to compare rate of convergence of these iterations to find out which ones converge faster to that fixed point .

Recently, several authors introduced different type iteration methods and they have proved that their iteration methods converges faster than Picard[12], Mann [9] and Ishikawa [7] iteration methods. In [5], Gürsoy concluded that Picard-S iteration method is faster than all Picard[12], Mann[9], Ishikawa [7], S [1], Noor [10] and SP [11] iterative methods, under contraction mappings.

In this paper, we show that a Picard-S iteration method [6] can be used to approximate fixed point of almost contraction mappings. Also, we provide an example to show that this point need not be unique. Furthermore, we present condition (B) [2], under which the uniqueness holds. Finally, we show that this iteration converges faster than modified SP iteration method [8] for the aforementioned class of mappings.

Throughout this paper the set of all positive integers and zero is shown by ℕ. Let B be a Banach space, C be a nonempty closed convex subset of B and T a self-map of C. An element u^* of C is called a fixed point of T if and only if Tu^*=u^* [3]. The set of all fixed point of T denoted by F_T.

1. Convergence Theorems

In this section we discuss the convergence of a Picard-S iterative scheme [6], which defined as follows:

$$\left\{\begin{array}{l} \qquad x_0 \in C \\ \quad x_{n+1} = Ty_n \\ y_n = (1-\beta_n)Tx_n + \beta_n Tz_n \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n \end{array}\right. \qquad (1)$$

Where $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ under the following kind of mappings.

Definition2.1 [3]: Let (B,‖. ‖) be a Banach space. A map T:B→B is called an Almost Contraction mapping if there exists a constant $\delta \in (0,1)$ and some $L \in (0,\infty)$ such that

$\|Tx - Ty\| \leq \delta\|x-y\| + L\|y - Tx\|$ for all x,y∈B (2)

Theorem2.2: Let T:C→C be an almost contraction map with $u^* \in F_T$ and $\{x_n\}_{n=0}^{\infty}$ an iterative sequence defined by (1) with real sequences $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ in [0,1] satisfying $\sum_{k=0}^{\infty}\beta_k\gamma_k = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point u^* of T.

Proof: Using Picard-S iterative scheme (1) and condition (2), we obtain:

$$\begin{aligned} \|z_n - u^*\| &\leq (1-\gamma_n)\|x_n - u^*\| + \gamma_n\|Tx_n - Tu^*\| \\ &\leq (1-\gamma_n)\|x_n - u^*\| + \gamma_n\delta\|x_n - u^*\| + \gamma_n L\|x_n - Tu^*\| \\ &\leq \left[1 - \gamma_n\big(1 - (\delta + L)\big)\right]\|x_n - u^*\| \qquad (3) \end{aligned}$$

Also

$$\begin{aligned} \|y_n - u^*\| &= \|(1-\beta_n)Tx_n + \beta_n Tz_n - u^*\| \\ &\leq (1-\beta_n)\|Tx_n - Tu^*\| + \beta_n\|Tz_n - Tu^*\| \\ &\leq (1-\beta_n)(\delta + L)\|x_n - u^*\| + \beta_n(\delta + L)\|z_n - u^*\| \qquad (4) \end{aligned}$$

Thus

$$\begin{aligned} \|x_{n+1} - u^*\| &= \|Ty_n - u^*\| \\ &= \|Ty_n - Tu^*\| \\ &\leq \delta\|y_n - u^*\| + L\|y_n - Tu^*\| \\ &= (\delta + L)\|y_n - u^*\| \qquad (5) \end{aligned}$$

Combining (3), (4) and (5), we get:

$$\begin{aligned} \|x_{n+1} - u^*\| &\leq (\delta + L)^2[(1-\beta_n)\|x_n - u^*\| + \beta_n[1 - \gamma_n(\delta + L)]\|x_n - u^*\|] \\ &= (\delta + L)^2\left[1 - \beta_n + \beta_n\left[1 - \gamma_n\big(1 - \ (\delta + L)\big)\right]\right]\|x_n - u^*\| \\ &= (\delta + L)^2\left[1 - \beta_n\gamma_n\big(1 - (\delta + L)\big)\right]\|x_n - u^*\| \end{aligned}$$

By induction

$$\begin{aligned} \|x_{n+1} - u^*\| &\leq (\delta + L)^{2(n+1)}\prod_{k=0}^{n}\left[1 - \beta_k\gamma_k\big(1 - (\delta + L)\big)\right]\|x_0 - u^*\| (6) \\ &\leq (\delta + L)^{2(n+1)}\|x_0 - u^*\| e^{-\left(1-(\delta+L)\right)\sum_{k=0}^{n}\beta_k\gamma_k} \end{aligned}$$

Since $\sum_{k=0}^{\infty}\beta_k\gamma_k = \infty$, $e^{-\left(1-(\delta+L)\right)\sum_{k=0}^{n}\beta_k\gamma_k}$→0 as n→∞, Which implies $\lim_{n\to\infty}\|x_n - u^*\| = 0$.

The following example illustrates that the fixed point of an almost contraction map need not be unique.

Example 2.3: Let T:[0,1] → [0,1] be the identity map. Then T satisfies condition (2) with $\delta \in (0,1)$ arbitrary and $L \geq 1 - \delta$. But note that the set of fixed points of T is the entire interval $[0,1]$.

The uniqueness will satisfy if the almost contraction map satisfying condition (B) converges to a unique fixed point using Picard-S iteration method. But first we need the following definition:

Definition2.4[2]: Let (B,‖. ‖) be a Banach space. A map T:B→B is said to satisfy condition (B) if there exist 0<δ<1 and L≥0 such that for all x,y∈B we have

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\min\{\|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\}$$

Theorem2.5: Let T:C→C be an almost contraction map satisfying condition (B) with $F_T \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ an iterative sequence defined by (1) with real sequences $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ in [0,1] satisfying $\sum_{k=0}^{\infty}\beta_k\gamma_k = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges to a unique fixed point u^* of T.

Proof: By theorem (2.2) , $\{x_n\}_{n=0}^{\infty}$ converges to $u^* \in F_T$.

We have to show that u^* is unique. Suppose that z^* is another fixed point of T. Therefore, we have

$$\begin{aligned} \|u^* - z^*\| &= \|Tu^* - Tz^*\| \leq \delta\|u^* - z^*\| + L\min\{\|u^* - Tu^*\|, \|z^* - Tz^*\|, \|u^* - Tz^*\|, \|z^* - Tu^*\|\} \\ &= \delta\|u^* - z^*\| \end{aligned}$$

But 0<δ<1, thus $\|u^* - z^*\| = 0$. Hence $u^* = z^*$.

Remark 2.6:
We notice in example (2.3) that T does not satisfy condition (B) since $|x-y| > \delta|x-y|$ for all $x \neq y$ and $0 < \delta < 1$. Therefore the fixed point is not unique.

2. The Rate of Convergence

In this section we compare the rate of convergence of Picard-S iteration with modified SP iterative scheme[8], which defined as:

$$\begin{cases} q_0 \in C \\ q_{n+1} = Tr_n \\ r_n = (1-\beta_n)s_n + \beta_n Ts_n \\ s_n = (1-\gamma_n)q_n + \gamma_n Tq_n \end{cases} \qquad (7)$$

where $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$.

Definition3.1 [4]: Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers with limits a and b respectively. Assume that there exists

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = l$$

i. If $l = 0$, then we say that $\{a_n\}_{n=0}^{\infty}$ converges faster to a than $\{b_n\}_{n=0}^{\infty}$ to b.
ii. If $0 < l < \infty$, then we say that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

The following theorem shows that the process (1) converges faster than process (8).

Theorem3.2: Let T:C→C be an almost contraction map satisfying condition (B) with $u^* \in F_T$, $\{q_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ are iterative sequences defined by modified SP (7) and Picard-S (1) iterative schemes respectively with real sequences $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ and a real number λ satisfying:

$0 < \lambda < \beta_n, \gamma_n < 1$ for all $n \in \mathbb{N}$

Then the Picard-S sequence converges faster than the modified SP iterative sequence to a unique fixed point of T, provided that the initial point is the same for both iterations.

Proof: By condition (B), the equation (6) will be:

$$\begin{aligned} \|x_{n+1} - u^*\| &\le \delta^2[1 - \beta_n\gamma_n(1-\delta)]\|x_n - u^*\| \\ &\le \delta^2[1 - \lambda^2(1-\delta)]\|x_n - u^*\| \\ &\vdots \\ \|x_{n+1} - u^*\| &\le \left[\delta^2[1-\lambda^2(1-\delta)]\right]^{n+1}\|x_0 - u^*\| \end{aligned}$$

Let $PS_{n+1} = \left[\delta^2[1-\lambda^2(1-\delta)]\right]^{n+1}\|x_0 - u^*\|$

Using (11), we obtain:

$$\begin{aligned} \|q_{n+1} - u^*\| &= \|Tr_n - u^*\| \\ &\le \delta\|r_n - u^*\| \\ &\quad + L\min\{\|r_n - Tr_n\|, \|u^* - Tu^*\|, \|r_n - Tu^*\|, \|u^* - Tr_n\|\} \\ &\le \delta\|r_n - u^*\| \qquad (8) \end{aligned}$$

Therefore

$$\begin{aligned} \|r_n - u^*\| &= \|(1-\beta_n)s_n + \beta_n Ts_n - u^*\| \\ &\le (1-\beta_n)\|s_n - u^*\| + \beta_n\|Ts_n - Tu^*\| \\ &\le [1-\beta_n(1-\delta)]\|s_n - u^*\| \qquad (9) \end{aligned}$$

Thus

$$\begin{aligned} \|s_n - u^*\| &= \|(1-\gamma_n)q_n + \gamma_n Tq_n - u^*\| \\ &\le [1-\gamma_n(1-\delta)]\|q_n - u^*\| + \\ &\qquad \gamma_n L\min\{\|q_n - Tq_n\|, \|u^* - Tu^*\|, \|q_n - Tu^*\|, \|u^* - Tq_n\|\} \\ &\le [1-\gamma_n(1-\delta)]\|q_n - u^*\| \qquad (10) \end{aligned}$$

Combining (8), (9) and (10), we get:

$$\begin{aligned} \|q_{n+1} - u^*\| &\le \delta[1-\beta_n(1-\delta)][1-\gamma_n(1-\delta)]\|q_n - u^*\| \le \delta[1-\lambda(1-\delta)]^2\|q_n - u^*\| \\ &\le \left[\delta[1-\lambda(1-\delta)]\right]^{n+1}\|q_0 - u^*\| \end{aligned}$$

Let $MS_{n+1} = \left[\delta[1-\lambda(1-\delta)]\right]^{n+1}\|q_0 - u^*\|$

$$\frac{PS_{n+1}}{MS_{n+1}} = \frac{\left[\delta^2[1-\lambda(1-\delta)]\right]^{n+1}\|x_0 - u^*\|}{[\delta[1-\lambda(1-\delta)]^2]^{n+1}\|q_0 - u^*\|} \to 0 \text{ as } n \to \infty.$$

Thus $\{x_n\}$ converges faster than $\{q_n\}$ to u^*.

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