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QUASI−SEMIPRIME SUBMODULES.

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Abstract

Let R be a commutative ring with unity and let N be a submodule of anon-zero leftR-moduleM, N is called semiprime if whenever $r^n x \in N$, $r \in R, x \in M, n \in Z^+$, implies $rx \in N$. In this paper we say that N is quasi−semiprimeiff$[N: M]$ is a semiprime ideal of R where $[N: M]$ =$\{r \in R: rM \subseteq$ N. We give many results of this type of submodules.

1. Introduction:-

A submodule of an R- module M which Dauns[1] was named semiprime submodules that they are generalized of semiprime ideals,which get big importance at last years, many studiesand searches are published about semiprimesubmodules by many people who care with the subject of commutative algebra and some of them are J. Dauns, R. L. McCasland, C.P.LU, P. F. Smith, M. E. Moore.The definition comes in [1] as following we say that a proper submodule N of an $R-$ module M is called semiprimesubmodule if wheneverr$^n x \in N$, $r \in R, x \in M, n \in Z^+$, implies $rx \in N$. Let's show the most important results that studies get. If N is a proper submoduleof an $R-$ module M, then N is semiprimeiff for each $r \in R$, $x \in M$ such that $r^2 x \in N$, then $rx \in N$,[2]. If N is a propersubmoduleof an $R-$ module M, then the following statements are equivalent: N is semiprime submodule of an $R-$ module M, then $[N: K]$ is semiprime ideal ofR where $[N: K] = \{r \in R: rK \subseteq N\}$[2], then $[N: \langle x\rangle]$ is semiprime ideal of R, $x \notin N$. If N is semiprime submodule, then $[N: M]$ is semiprime ideal of R[2], If $0 \neq M$ is Z-regular module, then every submodule is semiprime[2]. A proper submoduleN of an $R-$module M is said to be quasi−prime submodule of M if $[N: M]$ is a prime ideal of R, where $[N: M]$ =$\{r \in R: rM \subseteq N\}$[3]. In this paper we give the following: a proper submoduleN of an $R-$module M is called a quasi-semiprimesubmodulesiff$[N: M]$ is semiprime ideal of R.Let R be a commutative ring with identity and M is a non zero unite left R-module M, a proper submodule N of an $R-$module M is called semiprime if whenever $r^n x \in N$, $r \in R, x \in M, n \in Z^+$, implies $rx \in N$[2] equivalently N is called semiprime iff $r^2 x \in N, r \in R, x \in M$,implies $rx \in N$.Let N and K be two submodules of an $R-$module M, then, If $N \subsetneq K$, then $[N: M] \subseteq [K: M]$, If$N \subsetneq K$, then$[N: M] \subseteq [N: K]$. The set $\{r \in R: r^k \in I, \text{for some } k \in Z^+\}$ is called the radical ofI or the nil radical and denotedby $\sqrt{I}$,I is an ideal of a ringR. An ideal I of a ring R is said to be a semiprime ideal if $a^2 \in I$, for eacha $\in R$, then $a \in I$ equivalently if I is an ideal of a ring R, then I is semiprime ideal if and only if$I = \sqrt{I}$[4].

2. Quasi−Semiprime Submodules:-

Recall that a proper submodule N of an $R-$module M is called semiprimesubmodule if $N \neq M$ and wheneverr $\in R, m \in M$, $k \in Z^+$ such that $r^k m \in M$, thenrm $\in N$. [1]

Proposition (2.1) :[2]
If N is a proper submodule of an $R-$module M, then N is semiprimesubmodule of M if and only if whenever $r^2m \in N$, where $r \in R$, $m \in M$, then$rm \in N$.
Recall that a proper ideal I in a ringR is called semiprime ideal if$r^2 \in I$ implies that $r \in I$.[4]. IfM is a prime $R-$module, then $\mathrm{ann}_R N$ is a prime ideal for every non$-$zero submodule N of M , [3].A proper submoduleN of an $R-$module M is said to be a quasi$-$prime submodule of M if $[N: M]$ is a prime ideal of R, where $[N: M] =\{r \in R: rM \subseteq N\}$,[3].

We introduce the following:

Definition (2.2):-
A proper submoduleN of an$R-$module M is said to be a quasi$-$semiprimesubmodule of M if and only if $[N: M]$ is a semiprime ideal of R where $[N: M]= \{r \in R: rM \subseteq N\}$.

Remarks and examples(2.3):-
1. Every semiprimesubmodule of an $R-$module M is a quasi$-$semiprime submodule ofM, but the converse is not true ingeneral for example let $M = Z \oplus Z$ be a module over Zand N be a submodulegenerated by $(4,0)$, then $[N: M] = 0$ is a semiprime ideal ofZ, butN is not semiprime submodule of M.
2. Every quasi$-$prime submodule of an $R-$module M is a quasi$-$semiprime submoduleofM.
3. **Proof:** Let N be a quasi$-$prime submodule of an $R-$module M, thenN is a semiprimesubmodule[3], hence$[N: M]$ is a semiprime ideal of R,[2].Thus N is a quasi$-$semiprime submodule ofM, but the converse is not true ingeneral, for example the submodule$6Z$ of Z as $Z-$module is a quasi$-$semiprime submodule, but not quasi$-$prime submodule.
4. The submodule$\langle\bar{4}\rangle$ of the $Z-$module Z_8 is not a quasi$-$semiprime submodule since$[\langle\bar{4}\rangle: Z_8] = 4Z$ is not semiprime ideal of Z.
5. The submodule$6Z$ of the $Z-$module Z is a quasi$-$semiprime submodule of Z.
6. The intersection of any two quasi$-$semiprime submodules of an $R-$module M need not be quasi$-$semiprime submodule for example the$Z-$module Z_{12} has quasi$-$semiprime submodules let $N_1 = \langle\bar{2}\rangle$ and$N_2 = \langle\bar{3}\rangle$ be two quasi$-$semiprime submodules of Z_{12}, but $N_1 \cap N_2 = \langle\bar{0}\rangle$ is not quasi$-$semiprime submodule of Z_{12} since$[\langle\bar{0}\rangle: Z_{12}] = 12\,Z$ is not semiprime ideal of Z.

The following proposition shows that the concepts of aquasi$-$semiprime submodules and semiprimesubmodulesare equivalent in the class of multiplication modules.

Proposition (2.4) :-
Let N be a proper submodule of a multiplication $R-$module M. Then the following statements are equivalent:
1. Nis a quasi$-$semiprime submodule of M.
2. $[N: M]$is a semiprime ideal of R.
3. Nis a semiprimesubmodule ofM.

Proof: 1$\Longrightarrow$2 by definition of a quasi$-$semiprime submodules.
2$\Longrightarrow$ 3by a multiplication module .
3$\Longrightarrow$ 1 by Remark(2.3) (1).

Corollary (2.5):-
IfN is a quasi$-$semiprime submodule of a multiplication $R-$module M, then N is an intersection of some prime submodules.

Proof: SinceN is a quasi$-$semiprime submodule of a multiplication $R-$module M, then by aboveProposition N is a semiprimesubmodule, then $\sqrt{N} = N$,[5].

Definition (2.6) :[6]
AnR-module M is called $Z-$regular module if for each $m \in M$ there exists$f \in M^* = \mathrm{Hom}(M, R)$ such that $m = f(m).m$.

Proposition (2.7):-
Every submodule ofaZ $-$regular module is quasi$-$semiprime submodule.

Proof: Let M be aZ $-$regular module, then every submodule of M is semiprime [2], then by Remark(2.3) (1) every submodule ofM is quasi$-$semiprime submodule.

Definition (2.8) :[7]
An R $-$module M is called a multiplication module if for each submodule$N$ of$M$ there exists an ideal I of$R$ such that $N = IM$. In fact M is called a multiplication module if $[N: M]M = N$, for each submodule N of M.

Proposition (2.9):-
$N$is a quasi$-$semiprime submodule of a multiplication R $-$module M if and only if$I^2M \subseteq N$, for some ideal I of R, then $IM \subseteq N$.

Proof: Since$N$ is a quasi$-$semiprime submodule of a multiplication R $-$module M, then by Proposition(2.4)N is a semiprimesubmoduleand$I^2M \subseteq N$, then$IM \subseteq N$.For the converse, since $I^2M \subseteq N$, for some ideal I of Rimplies that$IM \subseteq N$, then N is semiprime submodule [5], then byRemark(2.3) (1)N is a quasi$-$semiprime submodule.

Proposition (2.10):-
If N is a proper submodule of a multiplication R $-$module M and $[N: M]$ is a primary ideal of R, then the following statements are equivalent:
1. $N$is a quasi$-$prime submodule of M.
2. $N$is a quasi$-$semiprime submodule of M.

Proof: 1⟹2 by Remark(2,3)(2).
2⟹1Since$N$ is a quasi$-$semiprime submodule of$M$, then$[N: M]$ is a semiprime ideal and by assumpation$[N: M]$ is a primary ideal of$R$, then$[N: M]$ is a prime ideal. Which implies that N is a quasi$-$prime submodule of M.

Recall that a submodule N of an R $-$module M is called injective envelope of N in M, denoted by$E_M(N)$,$E_M(N) = \{x = r\,m : r \in R, m \in M \text{ such that } r^k m \in N , k \in Z^+\}$.It is clear that $N \subseteq E_M(N)$[8].

Proposition (2.11):-
Let N be a proper submodule of a multiplication R $-$module M, then$E_M(N) = N$if and only if$N$is a quasi$-$semiprime submodule of M.

Proof: Since$E_M(N) = N$, then N is a semiprime submodule of M[9], then by Remark (2.3) (1) $N$is a quasi$-$semiprime submodule of M.For the converse, since$N$ isa quasi$-$semiprime submodule of M and M is a multiplication R $-$module, then by Proposition (2.4) N is semiprimesubmodule and hence$E_M(N) = N$.

If M and M are R $-$modules and$\emptyset: M \rightarrow M$ is an epimorphism with $ker\ \emptyset \subseteq N$. If$N$ is semiprimesubmodule in M, then $\emptyset(N)$issemiprime in $M$and If N' is a quasi$-$semiprime submodule in M and $ker\ \emptyset \ll M$, then $\emptyset^{-1}(N')$ is semiprime in M[2]. IfN is a quasi$-$semiprime submodule in$M$and if N' is a quasi$-$semiprime submodule in$M$.
Now, we have the following:

Proposition (2.12):-
IfM and$M$ are$R$ $-$modules and$\emptyset: M \rightarrow M$ is an epimorphism with $ker\ \emptyset \subseteq N$.If Nisa quasi$-$semiprimesubmodulein$M$, then $\emptyset(N)$is a quasi$-$semiprimesubmodule in M.

Proof: We want to show that$\emptyset(N)$ is a proper submoduleof M. Suppose not $\emptyset(N) = M$, then $\emptyset(N) = \emptyset(M)$, then $\exists\ m \in M$ such that$\emptyset(m) \in M' = \emptyset(N)$, $\exists n \in N$ such that $\emptyset(n) = \emptyset(m)$, hence $\emptyset(n - m) = 0$, then$n - m \in ker\ \emptyset \subseteq N$, then$N = M$(contradiction).
Now, we want to show that $\emptyset(N)$is a quasi$-$semiprime submodulein M. It is enough to prove $\sqrt{[\emptyset(N): M]} \subseteq [\emptyset(N): M]$. Let $x \in \sqrt{[\emptyset(N): M]}$, then $\exists\ n \in Z^+$ such that $x^n \in [\emptyset(N): M]$, then $x^n M \subseteq \emptyset(N)$, then$x^n m' = \emptyset(n), m' \in M$, $n \in N$. Since$\emptyset$ is onto, then $\exists\ m \in M$such that $\emptyset(m) = m'$, then $x^n \emptyset(m) = \emptyset(n)$, then $\emptyset(x^n m -$

$n) = 0$, thus $x^n m - n \in ker\ \emptyset \subseteq N$, then $x^n m \in N$, then $x^n M \subseteq N$, then $x^n \in [N:M]$, but $[N:M]$ is a semiprime ideal of R. Then $x \in [N:M]$, then $xM \subseteq N$, then $x\ m \in N$, then $\emptyset(x\ m) \in \emptyset(N)$, then $x\emptyset(m) \in \emptyset(N)$, then $x\ \emptyset(M) \subseteq \emptyset(N)$, then $x\ M \subseteq \emptyset(N)$, then $x \in [\emptyset(N):M]$.

Proposition (2.13):-
If M and M are $R-$modules and $\emptyset: M \to M$ is an epimorphism. If N' is a quasi$-$semiprime submodule in M, then $\emptyset^{-1}(N')$ is a quasi$-$ semiprime submodule in M.

Proof: It is clear that $\emptyset^{-1}(N') \subsetneq M$. We want to show that $\emptyset^{-1}(N')$ is a quasi$-$ semiprime submodule in M. It is enough to prove $\sqrt{[\emptyset^{-1}(N'):M]} \subseteq [\emptyset^{-1}(N'):M]$. Let $a \in \sqrt{[\emptyset^{-1}(N'):M]}$, then $\exists\ n \in Z^+$ such that $a^n \in [\emptyset^{-1}(N'):M]$, then $a^n M \subseteq \emptyset^{-1}(N')$, then $\emptyset(a^n)M \subseteq N'$, then $(\emptyset(a))^n\ M \subseteq N'$, then $(\emptyset(a))^n \in [N':M]$. Since N' is a quasi$-$semiprime submodule in M, then $[N':M]$ is a semiprime ideal of R. Then $\emptyset(a) \in [N':M]$, then $\emptyset(a)M \subseteq N'$, then $a\ M \subseteq \emptyset^{-1}(N')$, then $a \in [\emptyset^{-1}(N'):M]$. Which implies that $\emptyset^{-1}(N')$ is a quasi$-$ semiprime submodule in M.

Proposition (2.14):-
Let K be a proper submodule of an $R-$module M and N be a quasi$-$semiprime submodule of M with $N \subseteq K$ such that $[N:M]$ is a maximal ideal of R, then K is a quasi$-$semiprime submodule of M.

Proof: We want to show that K is a quasi$-$semiprime submodule of M. It is enough to prove $\sqrt{[K:M]} \subseteq [K:M]$. Let $a \in \sqrt{[K:M]}$, then $\exists\ n \in Z^+$ such that $a^n \in [K:M]$, then $a^n M \subseteq K$. Since $N \subseteq K$, then $[N:M] \subseteq [K:M]$, but K be a proper submodule of M, then $[K:M]$ is a proper ideal of R. Since $[N:M]$ is a maximal ideal of R, then $[N:M] = [K:M]$. Thus $a^n \in [K:M] = [N:M]$, then $a^n \in [N:M]$, but $[N:M]$ is a, semiprime ideal of R, then $a \in [N:M] = [K:M]$, then $a \in [K:M]$.
which implies that K is a quasi$-$ semiprime submodule in M.

Proposition (2.15):-
Let K be a proper submodule of an $R-$module M and N be a quasi$-$semiprime submodule of M with $[N:M] = [K:M]$, then K is a quasi$-$semiprime submodule of M.

Proof: We want to show that $[K:M]$ is a semiprime ideal of R. Since N is a quasi$-$semiprime, then $[N:M]$ is semiprime ideal and $[N:M] = [K:M]$, then $[K:M]$ is semiprime ideal, then K is a quasi$-$semiprime submodule of M.

Proposition (2.16):-
Let N be a quasi$-$semiprime submodule of an $R-$module M and K be a semiprime submodule of M, then $N \cap K$ is a quasi$-$semiprime submodule of M.

Proof: We want to show that $[N \cap K:M]$ is a semiprime ideal of R. It is enough to prove $\sqrt{[N \cap K:M]} \subseteq [N \cap K:M]$. Let $x \in \sqrt{[N \cap K:M]}$, then $\exists\ n \in Z^+$ such that $x^n \in [N \cap K:M] \subseteq [N:M]$, but N is a quasi$-$semiprime submodule of M, then $[N:M]$ is a semiprime ideal of R. Thus $x \in [N:M]$, then $x\ M \subseteq N$. On the other hand $x^n \in [N \cap K:M] \subseteq [K:M]$, then $x\,M \subseteq K$, then $x\,M \subseteq N \cap K$, then $x \in [N \cap K : M]$. Which implies that $N \cap K$ is a quasi$-$ semiprime submodule in M.

Proposition (2.17):-
Let N be a quasi$-$semiprime submodule of an $R-$module M and K be any submodule of M such that $K \nsubseteq N$, then $N \cap K$ is a quasi$-$semiprime submodule of K.

Proof: Since $K \nsubseteq N$, then $N \cap K$ is a proper submodule of K. Let $r \in R$ such that $r^2 \in [N \cap K:K]$. To prove $r \in [N \cap K:K]$. Since $r^2\ m \in N \cap K$ for each $m \in M$, then $r^2\ m \in N$, but N is a quasi$-$semiprime submodule of M, then $r\ m \in N$ and $m \in K$, then $r\ m \in N \cap K$, for each $m \in K$, then $r \in [N \cap K:K]$. Which implies that $N \cap K$ is a quasi$-$ semiprime submodule in K.

3. Quasi−Semiprime modules.

Recall that an R −module M is said to be a semiprime module if(0) is a semiprimesubmodule of M [10].

An R −module M is called a quasi−prime module iff$ann_R N$ is a prime ideal for each non−zero submodule N of M,[3].An R −module M is a quasi−prime module iff(0) is a quasi−prime submodule of$M$,[3].

Compare the following with (1.1.3)(13),[3].

Proposition (3.1):-
Let M be an R −module, if M is asemiprime$R$ −module, then $ann_R N$ is a semiprime ideal for every non−zerosubmodule$N$of M.

Proof: Let $r \in R$ such that $r^n \in ann_R N$, $n \in Z^+$.We want to show that $r \in ann_R N$.Since$r^n \in ann_R N$, then $r^n x = 0$ for all $x \in N$, then$r^n x \in (0)$ and M is a semiprime module, then $rx \in (0)$, then $r \in ann_R N$. Which implies that $ann_R N$ is a semiprime ideal.

Now, we introduce the following:

Definition (3.2):-
An R −module M is said to be quasi−semiprime module if and only if $ann_R N$ is a semiprime ideal of R for every non−zero submodule N of M.

Remarks and examples(3.3):-
1. Every semiprime$R$ −module is a quasi−semiprime module .
2. The proof follows directly by prop.(3.1), but the converse is not true in general for example Z_4 as Z −module is a quasi−semiprime module since $ann_Z N$ is a semiprime ideal of Z for each submodule N of M, but (0) is not semiprime submodule of Z_4 .
3. $Z_{P^\infty}$is not quasi−semiprime Z −module .
4. **Proof:** Every submodule of$Z_{P^\infty}$is the form $\left(\frac{1}{P^n}+Z\right)$, $n \in Z^+ ann_Z\left(\frac{1}{P^n}+Z\right) = P^n Z$ is not a semiprime ideal of Z.
5. The homomorphic image of a quasi−semiprime module need not be quasi−semiprime module for example it is clear that the Z as Z −module is a quasi−semiprime module, but $\frac{Z}{12Z} \cong Z_{12}$ is not quasi−semiprime$Z$ −module.

Proposition (3.4):-
Let Mbe an R −module, if M is a semiprime$R$ −module, then M is a quasi−semiprime module.

Proof: Since M is a semiprime$R$ −module, then by prop. (3.1) $ann_R N$ is semiprime ideal of R and by definition (3.2), then$M$ is a quasi−semiprime module.

Compare the following with Prop. (2.2.1), [3].

Theorem(3,5):-
Let Mbe an R −module, then M is a quasi−semiprime moduleiff(0) is a quasi−semiprimesubmoduleof$M$.

Proof: Suppose that M is a quasi−semiprime module.We want to show that (0) is a quasi−semiprime submodule . Since M is a quasi−semiprime module, then $ann_R M$ is a semiprime ideal of R, but $ann_R M = [(0): M]$, then by definition (2.2)(0) is a quasi−semiprimesubmodule of$M$. Conversely, if (0) is a quasi−semiprime submodule of$M$.To prove M is a quasi−semiprime module. Since (0) is a quasi−semiprime submodule, then definition (2.2)implies that $[(0): M]$ is a semiprime idealof R, but$ann_R M = [(0): M]$, then M is aquasi−semiprime module.

Corollary(3,6):-
Let$N$ be a proper submodule of an R −module M, then N is a quasi−semiprime submodule of$M$ iff $M/_N$ is a quasi−semiprime module.

Proposition (3.6):-
Every direct summand of a quasi−semiprime R −module is aquasi−semiprime submodule.

Proof: Suppose that N_1 and N_2 are two submodules ofan R −module M such that $M = N_1 \oplus N_2$ and M is a quasi−semiprime R −module.To prove N_1 is a quasi−semiprime submodule.Let $r \in R$, $m \in M$ such that $r^2 \in [N_1 : M]$, but $m \in M = N_1 \oplus N_2$ implies that there exists $m_1 \in N_1, m_2 \in N_2$ such that $m = m_1 + m_2$.
Now, $r^2 m \in N_1$, then $r^2(m_1 + m_2) = (r^2 m_1 + r^2 m_2) \in N_1$, then $r^2 m_2 \in N_1 \cap N_2 = (0) = ann_R M$ .Since M is quasi−semiprime R −module, then $ann_R M$ is a semiprime ideal of R.Then $rm_2 \in ann_R M = (0)$, so $rm_2 = (0)$, thus $rm = r(m_1 + m_2) = rm_1 + rm_2 = rm_1 \in N_1$, then $rm \in N_1$.Which implies that N_1 is a quasi−semiprime submodule ofM.
Similarly N_2 is a quasi−semiprimesubmoduleof M.

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