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RESEARCH ARTICLE

ON NONDIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING INVOLVING TYPE-I α -INVEX FUNCTIONS.

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Abstract

The aim of this paper is to study a nondifferentialblemultiobjective programming problem with inequality constraints. In this paper we introduce the concept to type-I α-invex, weak strictly pseudo-quasi type-I α-invex, strong pseudo-quasi type-I α-invex, weak quasi-strictly-pseudo type-I α-invex and weak strictly-pseudo type-I α-invex functions. By utilizing these new notions we derive a Fritz John type sufficient optimality condition and establish Mond-Weir type and Mond-Weir type duality results for the nondiffferentiablemultiobjective programming problem.

Introduction:-

Convexity plays a vital role in many aspects of mathematical programming (see, for example, Bazaraa et al. [3] and Mangasarian [12] In order to study the optimization problems in a wider context various useful generalizations of the notion of convexity have been introduced. Hanson [8] introduced the class of invex functions. Later, Hanson and Mond [9] defined two new classes of functions called type-I and type - II function. This concept was extended by Rueda and Hasnon [29] to pseudo-type-I and quasi-type-I functions. Univex functions were introduced and studied by Bector et al. [4]. Rueda *et al.* [30] studied optimality and duality results for several mathematical programs by combining

The concept of type-I and univex functions. Kaul*et al.* [11] considered a multiple objective problem with type-I functions and obtained some results on optimality and duality. Mishra [15] studied a multiple objective nonlinear programming problem by combining the concept of type-I, pseudo-type-I, quasi-type-I, quasi-pseudo-type-I, pseudo-quasi-type-I and univex functions. More details on type-I functions can be found in Ye [33], Suneja and Srivastava [31], Mishra *et al.* [19, 21, 22] and Mishra *et al.* [23, 24]. Aghezzaf and Hachimi [1] introduced new class of generalized type-I vector valued functions and derived various duality results for a nonlinear multiobjective programming problem.

Theoretical problems of differentiable programming can be solved by substituting invexity for convexity e.g. Hanson [8], Craven [5], Egudo and Hanson [7], and Jayakumar and Mond [10]. But corresponding conclusion cannot be obtained in nondifferentiable programming with the aid of invexity introduced by Hanson [8] because the existence of a derivative is required in the definition of invexity.
Generalization of invexity to locally Lipschitz functions, with derivative replaced by Clarke generalized gradient has been considered by Craven [6], Reiland [28], Mishra and Mukherjee [17], Mishra [13, 14], and Mishra and

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Giorgi[16]. However, Antczak [2] used directional derivative, in association with a hypothesis of an invex kind, following Ye [33].

Noor [26] and Mishra and Noor [18] have studied some properties of the α-preinvex functions and their differentials. Recently Mishra, Pant and Rautela [20] and Pant and Rautela [27] introduced the concepts of strict pseudo α-invex, quasi α-invex, weak strictly pseudo quasi α-invex, strong pseudo quasi α-invex, weak quasi strictly pseudo α-invex and weak strictly pseudo α-invex functions.

In the present paper, as an application of the new classes of type-I α-invex functions we have considerd a nondifferentiablemultiobjective programming problem and derive Fitz John type sufficient optimality conditions for a (weakly) pareto efficient solution to the problem. Further the Mond-Weir type and general Mond-Weir type of duality results are also obtained.

Preliminaries:-

Throughout this paper, we will use the following conventions for vectors in R^n:

$$x = y \Leftrightarrow x_i = y_i, \quad i = 1, 2, ..., n$$

$$x > y \Leftrightarrow x_i \geq y_i, \quad i = 1, 2, ..., n$$

$$x \geq y \Leftrightarrow x_i \geqq y_i, \ i = 1, 2, ..., n \text{ but } x \neq y.$$

Let X be a nonempty subset of R^n, $\eta : X \times X \times X \to R^n$ is an n-dimensional vector valued function and α (x, y, z): $X \times X \times X \to R_+$ be a trifunction.

First, we recall some known results and concepts.

Definition 2.1: A subset $X \subseteq R^n$ is said to be α-invex set, if there exist $\eta : X \times X \times X \to R^n$ and $\alpha(x,u,v) : X \times X \times X \to R_+$ such that for all $x \in X$,

$$u + \lambda\alpha(x,u,v)\eta(x,u,v) \in X$$

$$v + \lambda\alpha(x,u,v)\eta(x,u,v) \in X, x, u, v \in X, \lambda \in [0,1]$$

Note that α-invex set need not to be convex set.

The following example from Noor (2004) shows that α-invex set need not to be convex set.

Example 2.1: The set $X = R - \left(-\frac{1}{2}, \frac{1}{2}\right)$ is an invex set with respect to α (x, u, v) = 1 and η, where

$$\eta = \begin{cases} x - u - v & \text{for } x > 0, \quad u > 0, v > 0 \\ u - x - v & \text{for } u > 0, \quad x < 0, v < 0 \\ v - x - u & \text{for } v > 0, \quad x < 0, u < 0 \end{cases}$$

It is clear that X is not a convex set.

From now onwards we assume that the set X is a nonempty α-invex set with respect to α (x, u, v) and η (x, u, v). unless otherwise specified.

Definition 2.2: The function $f : X \subseteq R^n \to R^k$ on the α-invex set is said to be α-pre-invex function if there exist $\eta : X \times X \times X \to R^n$ and $\alpha(x,u,v) : X \times X \times X \to R_+$

such that for all $x \in X$,

$$f\left(x + \lambda\alpha\right)\left(x,u,v)\lambda(x,u,v)\right) \leq (1-\lambda) f(u,v) + \lambda f(x), \forall x, u, v \in X \text{ and } \lambda \in [0,1]$$

We consider the following mathematical programming problem.

<P> Minimize f(x): subject to $g(x) \leq 0, x \in X$,

where $f : X \subseteq R^n \to R^k$ and $g : X \subseteq R^n \to R^m$ are functions on a set $X \subseteq R^n$ (a nonempty α-invex set)

Throughout this paper we use the notation.

$$\alpha(x,u,v)f'(u,\eta(x,u,v)) = \lim_{\lambda \to 0_+} \frac{f(u+\lambda\alpha(x,u,v)\eta(x,u,v)) - f(u) - f(v)}{\lambda}$$

and, a similar natation for $\alpha(x,u,v)g'(u,\eta(x,u,v))$

and $\alpha(x,u,v)g'(v,\eta(x,u,v))$

Let D be a nonempty α-invex set such that, D = {x ∈ X : g(x) ≤ 0} is the set of it all the feasible solution for (P) and denote I = {1, 2, ...k,}, M = {1, 2, ..., m}, $J(x) = \{j \in M : g_j(x) = 0\}$ and $\bar{J}(x) = \{j \in M : g_j(x) < 0\}$.

Now, this implies $J(x) \cup \bar{J}(x) = M$

Definition 2.3: The pair (f, g) is said to be type I α-invex with respect to α and η at u, v ∈ X if there exist functions $\alpha(x,u,v) : X \times X \times X \to R_+$ and $\eta : X \times X \times X \to R^n$ such that,

$$f(x) - f(u) \geqq \alpha(x,u,v)f'(u,\eta(x,u,v))$$

$$-g(u) \geqq \alpha(x,u,v)g'(u,\eta(x,u,v)) \ \forall x,u,v \in X$$

$$-g(v) \geqq \alpha(x,u,v)g'(v,\eta(x,u,v)) \ \forall x,u,v \in X$$

Definition 2.4: The pair (f, g) is said to be weak strictly pseudo-quasi type-I α-invex with respect to α and η at u, v ∈ X, if there exists function $\alpha(x,u,v) : X \times X \times X \to R_+$ and

$\eta : X \times X \times X \to R^n$ such that,

$$f(x) - f(u) \le 0 \Rightarrow \alpha(x,u,v)f'(u,\eta(x,u,v)) < 0$$

$$f(x) - f(v) \le 0 \Rightarrow \alpha(x,u,v)f'(v,\eta(x,u,v)) < 0$$

$$-g(u) \le 0 \Rightarrow \alpha(x,u,v)g'(u, \eta(x,u,v)) \leqq 0$$

$$-g(v) \le 0 \Rightarrow \alpha(x,u,v)g'(v,\eta(x,u,v)) \leqq 0 \ \forall x,u,v \in X$$

Definition 2.5: The pair (f, g) is said to be strong pseudo-quasi type-I α-invex with respect to α and η at u, v ∈ X. If there exist function

$\alpha(x,u,v) : X \times X \times X \to R_+$ and $\eta : X \times X \times X \to R^n$,

such that,

$$f(x) - f(u) \le 0 \Rightarrow \alpha(x,u,v)f'(u,\eta(x,u,v)) \le 0$$

$$f(x) - f(v) \le 0 \Rightarrow \alpha(x,u,v)f'(v,\eta(x,u,v)) \le 0$$

$$-g(u) \le 0 \Rightarrow \alpha(x,u,v)g'(u,\eta(x,u,v)) \le 0$$

$$-g(v) \le 0 \Rightarrow \alpha(x,u,v)g'(v,\eta(x,u,v)) \leqq 0 \ \forall x,u,v \in X$$

Example 2.2: Consider the function $f = (f_1, f_2, f_3) : [-2, 3] \to R$

$$f_1 = \begin{cases} 0, & -2 \le x \le 1 \\ x^2, & 1 \le x \le 3 \end{cases}$$

$$f_2 = \begin{cases} x^3, & -2 \le x \le 1 \\ 5, & 1 \le x \le 3 \end{cases}$$

$$f_3 = \begin{cases} 6, & -2 \le x \le 1 \\ 3x^2 - 4, & 1 \le x \le 3 \end{cases}$$

And the function $g = (g_1, g_2, g_3) : [-2, 3] \to R$ defined by

$$g_1 = \begin{cases} -x^2, & -2 \le x \le 1 \\ -4, & 1 \le x \le 3 \end{cases}$$

$$g_2 = \begin{cases} 5x, & -2 \le x \le 1 \\ x^4 - 6, & 1 \le x \le 3 \end{cases}$$

$$g_3 = \begin{cases} 0, & -2 \le x \le 1 \\ 2x^2 - 8, & 1 \le x \le 3 \end{cases}$$

Clearly f_1, f_2, f_3, g_1, g_2, g_3 are not differentiable function at $x = 1$. The feasible region is nonempty, Let α (x, u, v) = 1

$\eta(x,u,v) = \dfrac{x^2(x-u-v)}{2}$ and u = 2, v = 2

(i) If $x \in [-2, 1]$ and $f_1(x) + f_2(x) + f_3(x) \le f_1(1) + f_2(1) + f_3(1)$

i.e., $0 + x^3 + 0 \le 1 + 1 + 6$

$\Rightarrow x^3 \le 8$

$\Rightarrow x \le 2$

which further implies that

$\alpha(x,u,v)f_1'(u,\eta(x,u,v)) + \alpha(x,u,v)f_2'(u,\eta(x,u,v))$

$+\alpha(x,u,v)f_3'(u,\eta(x,u,v)) = \dfrac{12x^2(x-4)}{2} = 6(x^2(x-4))$

and $-g_1(u) - g_2(u) - g_3(u) \le 0$, which implies that

$$\alpha(x,u,v)g_1'(u,\eta(x,u,v)) + \alpha(x,u,v)g_2'(u,\eta(x,u,v)) + \alpha(x,u,v)g_3'(u,\eta(x,u,v)) < 0$$

(ii)The case $x \in [1, 3]$ can be verified. similarly.

Thus, (f, g) is strong pseudo-quasi type-I α-invex with respect to α and η at x = 1

How ever (f, g) is not type –1 α-invex with respect to same α and η at x = 1.

Definition 2.6: The pair (f, g) is said to be weak quasi-strictly-pseudo type-I α-invex with respect to α and η at u, v $\in$ X, if there exist function $\alpha(x,u,v): X \times X \times X \to R_+$ and $\eta: X \times X \times X \to R^n$ such that,

$f(x) - f(u) \le 0 \Rightarrow \alpha(x,u,v)f'(u,\eta(x,u,v)) \leqq 0, \forall x,u,v \in X$

$f(x) - f(v) \le 0 \Rightarrow \alpha(x,u,v)f'(v,\eta(x,u,v)) \le 0, \forall x,u,v \in X$

$-g(u) \leqq 0 \Rightarrow \alpha(x,u,v)g'(u,\eta(x,u,v)) \leqq 0, \forall x,u,v \in X$

$-g(v) \leqq 0 \Rightarrow \alpha(x,u,v)g'(v,\eta(x,u,v)) \leqq 0, \forall x,u,v \in X$

Definition 2.7:

The pair (f, g) is said to be weak strictly-pseudo type-I α-invex with respect to α and η at u, v $\in$ X, if there exist function

$\alpha(x,u,v): X \times X \times X \to R_+$ and $\eta: X \times X \times X \to R^n$

such that,

$f(x) - f(u) \le 0 \Rightarrow \alpha(x,u,v)f'(u,\eta(x,u,v)) < 0, \forall x,u,v \in X$

$f(x) - f(v) \le 0 \Rightarrow \alpha(x,u,v)f'(v,\eta(x,u,v)) < 0, \forall x,u,v \in X$

$-g(u) \leqq 0 \Rightarrow \alpha(x,u,v)g'(u,\eta(x,u,v)) < 0, \forall x,u,v \in X$

$-g(v) \leqq 0 \Rightarrow \alpha(x,u,v)g'(v,\eta(x,u,v)) < 0, \forall x,u,v \in X$

Definition 2.8:

A point $u \in D$ is said to be a weak pareto efficient solution for (P) it the relation $f(u) < f(x)$ holds for all $x \in D$.

Definition 2.9: A point $u \in$ □D is said to be a locally weak pareto efficient solution for (P) if there exist a neighborhood N (u) around u such that f(u) < f(x), holds □□x, $\forall x \in N(u) \cap D$.

The following results from Antezak (2002) and Weir and Mond (1988) type will be needed in the next section.

Lemma 2.1: If u is a locally weak pareto or a weak pareto efficient solution of (P) and if g_j is continuous at u for if $j \in \bar{J}\ (u)$ then the following system of inequalities.

$$f'(u, \eta(x, u, v)) < 0$$

$$g'_{J(u)}(u, \eta(x, u, v)) < 0$$

has no solution for $x \in$ □X

Definition 2.10: A function g is said to satisfy the generalized slaters constraint qualification at u $\in D$ if g is α-invex at u and there exist $u \in D$ such that,

$$g_j(u) < 0,\ j \in J(u)$$

Lemma 2.2: (Fritz John type necessary optimality condition).

Let x be a weak parato efficient solution for (P).

More over we assume that g_j is continuous for $j \in \bar{J}(u)$, f and g are directionally differentiable at u with $f'(u, \eta(x, u, v))$ and $g'_{J(u)}(u, \eta(x, u, v))$ □□ α-preinvex functions of x on X. Moreover, we assume that g satisfies the generalized slaters constraint qualification at u.

Then there exist $\bar{\xi} \in R^k_+, \bar{\mu} \in R^m_+, \bar{\tau} \in R^m_+$ such that $(u, \bar{\xi}, \bar{\mu}, \bar{\tau})$ satisfies the following conditions,

$$\bar{\xi}^T f'(u, \eta(x, u, v) + (\bar{\mu}^T + \bar{\tau}^T) g'_{J(u)}(u, \eta(x, u, v)) \geq 0, \forall x \in x \quad (2.1)$$

$$\left(\bar{\mu}^T + \bar{\tau}^T\right) g(u) = 0, \quad (2.2)$$

$$g(u) \leqq 0 \quad (2.3)$$

Sufficient Optimality Conditions:

In this section, we establish a Fritz John type sufficient optimality condition.

Theorem 3.1:

Let u be a feasible solution for (P) at which conditions (1) – (3) are satisfied. Moreover, it any one if the following conditions is satisfied.

(a) $\left(\bar{\xi}^T f, \left(\bar{\mu}^T + \bar{\tau}^T\right) g\right)$ is strong pseudo-quasi type-1 α-invex at u with respect to some α_0, α_1 and η;

(b) $\left(\bar{\xi}^T f, \left(\bar{\mu}^T + \bar{\tau}^T\right) g\right)$ is weak strictly pseudo-quesi type-1 α-invex at u with respect to some α_0, α_1 and η;

(c) $\left(\bar{\xi}^T f, \left(\bar{\mu}^T + \bar{\tau}^T\right) g\right)$ is weak strictly pseudo type-1 α-invex at u with respect to some α_0, α_1 and η;

then u is a weak pareto efficient solution for (P).

Profit: We prove the theorem by contradiction.

Let us assume that u is not a weak pareto efficient solution of (P). Then there is a feasible solution x of (P) such that,

$f_i(x) < f_i(u)$ for any i = 1, 2, ..., k

$$\Rightarrow f_i(x) - f_i(u) < 0$$

$$\Rightarrow \bar{\xi}^T f_i(x) - \bar{\xi}^T f_i(u) < 0, \quad (\text{Since } \bar{\xi}^T > 0)$$

Now from the feasibility of x and (2.2), we get

$$\left(\bar{\mu}^T + \bar{\tau}^T\right)g(x) - \left(\bar{\mu}^T + \bar{\tau}^T\right)g(u) \le 0$$

If the condition (a) is satisfied, then from the above two inequalities, we get,

$$\bar{\xi}^T \alpha_0(x,u,v) f'(u, \eta(x,u,v)) < 0$$

and $(\bar{\mu}^T + \bar{\tau}^T)\alpha_1(x,u,v) g'(u,\eta(x,u,v)) \leqq 0$

By the positivity of α_0 and α_1 and above two inequalities reduces to

$$\xi^{-T} f'(u,\eta(x,u,v)) < 0 \text{ and } \left(\bar{\mu}^T + \bar{\tau}^T\right) g'(u,\eta(x,u,v)) \leqq 0$$

From above two inequalities, we get,

$$\bar{\xi}^T f'(u,\eta(x,u,v)) + \left(\bar{\mu}^T + \bar{\tau}^T\right) g'(u,\eta(x,u,v)) < 0$$

This condradicts (2.1)

If condition (b) is satisfied, we assume that ‘u’ is not a weak pareto efficient solution of (P). Then there is a feasible solution x of (P)

Such that, $f_i(x) - f_i(u) < 0$

$$\Rightarrow \bar{\xi}^T f_i(x) - \bar{\xi}^T f_i(u) < 0, \text{ (since } \bar{\xi}^T > 0\text{)}$$

Now by condition (b) and (2.2) we get

$$\bar{\xi}^T \alpha_0(x,u,v) f'(u,\eta(x,u,v)) < 0$$

$$\text{and} \left(\bar{\mu}^T + \bar{\tau}^T\right) \alpha_1(x,u,v) g'(u,\eta(x,u,v)) < 0$$

by the positivity of α_0 and α_1 the above two inequalities reduces to

$$\bar{\xi}^T f'(u,\eta(x,u,v)) < 0$$

$$\text{and} \left(\bar{\mu}^T + \bar{\tau}^T\right) g'(u,\eta(x,u,v)) < 0$$

From the above two inequalities, we get,

$$\bar{\xi}^T f'(u,\eta(u,v)) + \left(\bar{\mu}^T + \bar{\tau}^T\right) g'(u,\eta(x,u,v)) < 0$$

This is again a contradiction to (2.1)

Now for the part (c), following the similar process,

we get, $\bar{\xi}^T f'(u,\eta(x,u,v)) + \left(\bar{\mu}^T + \bar{\tau}^T\right) g'(u,\eta(x,u,v)) < 0$, which also contradicts (2.1) and completes the proof.

Example 3.1: Consider function $f = (f_1, f_2, f_3)$ defined on X = R,

by $f_1(x) = x^2$, $f_2(x) = x^3$, $f_3(x) = x^4$ and function g defined on X = R by

$$g = \begin{cases} -2x^2, & -1 \le x \le 2 \\ -x^3, & 2 \le x \le 3 \\ -2x^4, & 3 \le x \le 3.5 \end{cases}$$

Clearly, g is not differentiable at x = 2, but only directionally differentiable at x = 2. The feasible region is non empty.

Let, α (x, u, v) = 1

$$\eta(x,u,v) = \frac{x-u-v}{2}, \text{ and } u = 0, v = 0$$

(i) If $x \in [-1, 2]$, $-g(u) = 0$, $-g(v) = 0$ implies that $\alpha(x, u, v)\, g'(x, u, v) = 0$

(ii) The case $x \in [2, 3.5]$ can be verified similarly

$$f(x) \leqq f(u) \Rightarrow \alpha(x,u,v) f'(u,\eta(x,u,v)) = 0, \text{ for all } x$$

Thus (f, g) is strong pseudo-quasi type-1 α -invex at x = 0.
But (f, g) is not type-I α -invex at x = 0, with respect to α (x, u, v) = 1
and η (x, u, v) $= \dfrac{x-u-v}{2}$,
Then by theorem 3.1(a), u is a weak pareto efficient solution for the given multiobjective programming problem.

Mond-Weir Duality: Now in relation to (P). We consider the dual problem in the format of Mond Weir (1981).
(MWD) Maximize f(y) = {$f_1(y)$, $f_2(y)$, ...,$f_k(y)$}, subject to

$$\xi^T f' + \left(u^T + \tau^T\right) g'\left(y, \eta(x, y)\right) > 0, \text{ of all } x \in D \quad (4.1)$$

$$\left(\mu_j + \tau_j\right) g_j(y) \geq 0, \; j = \{1, 2, 3, ..., m\} \quad (4.2)$$

$$\xi^T e = 1 \quad (4.3)$$

$$\xi \in R^k_+, \mu \in R^m_+, \tau \in R^m_+$$

where e = (1, 1, ..., 1) $\in R^k$

Let $$W = \begin{cases} (y, \xi, \mu, \tau) \in X \times R^k \times R^m : \xi^T f'(y, \eta(x, y)) \\ \quad + (\mu^T + \tau^T) g'(y, \eta(x, y)) \geq 0 \\ (\mu_j + \tau_j) g_j(y) \geq 0, j = 1, 2, ..., m \\ \xi \in R^k_+, \mu \in R^m_+, \tau \in R^m_+, \xi^T e = 1 \end{cases}$$

denote the set of all feasible solution of (MWD), we also denote by $Pr_X W$ the projection of set W on X.

Theorem 4.1: (Weak Duality)

Let x and (y, ξ, μ, τ) be feasible solution for (P) and (MWD) respectively. Moreover, we assume that one of the following conditions holds.

(a) $(f_+, \bar{\mu}^T + \bar{\tau}^T, g)$ is strong pseudo-quasi type-I α -invex at y on $D \cup pr_x w$ with respect to some α_0, α_1 and η.

(b) $(f, \bar{\mu}^T + \bar{\tau}^T, g)$ is weak strictly pseudo-quasi type-1 α -invex at y on $D \cup pr_x w$ with respect to α_0, α_1 and η.

(c) $(f, \bar{\mu}^T + \bar{\tau}^T, g)$ is weak strictly pseudo-quasi type-1 α -invex at y on $D \cup pr_x w$ with respect to some α_0, α_1 and η.

Then the following can not hold:
$f(x) \leq f(y)$

Proof: Suppose that

$$f(x) \leq f(y), \text{ i.e., } f(x) - f(y) \leq 0 \quad (4.4)$$

Since x is feasible for (P) and (y, ξ, μ, τ) is feasible for (MWD). It follows that,

$$-\sum_{j=1}^{m} (\mu_j + \tau_j) g_j(y) \leq 0 \quad (4.5)$$

If condition (a) is satisfied, (4.4) and (4.5) imply

$$\alpha_0(x, y) f'(y, \eta(x, y)) \leq 0, \; \sum_{j=1}^{m} (\mu_j + \tau_j) \alpha_1(x, y) g'(y, \eta(x, y)) \leq 0$$

By the positivity of α_0 and α_1 and the above two inequatities reduce to.

$$f'\left(y, \eta(x, y)\right) \leq 0 \quad (4.6)$$

and $\sum_{j=1}^{m}(\mu_j+\tau_j)g'(y,\eta(x,y))\leq 0$ (4.7)

Since $\xi \geq 0$ from (4.6) and (4.7) we get,

$$\sum_{i=1}^{k}\xi_i f_i'(y,\eta(x,y))+\sum_{i=1}^{m}(\mu_j+\tau_j)g_j'(y,\eta(x,y))<0 \quad (4.8)$$

This contradicts (4.1). Hence the assertion.
If the condition (b) is satisfied, from (4.4) and (4.5) we get,

$$\alpha_0(x,y)f'(y,\eta\ (x,y)<0,\sum_{j=1}^{m}\left(\mu_j+\tau_j\right)\alpha_1(x,y)g'(y,\eta\ (x,y))\leqq 0$$

By the positivity of α_0 and α_1 the above inequalities reduce to,

$$f'(y,\eta(x,y))<0 \quad (4.9)$$

$$\sum_{j=1}^{m}\left(\mu_j+\tau_j\right)g'(y,\eta(x,y))\leqq 0, \quad (4.10)$$

Since $\xi \geq 0$ (4.9) and (4.10) imply (4.8) again a contradiction to (4.1).

If the condition (c) is satisfied from (4.4) and (4.5) we get,

$$\alpha_0(x,y)f'(y,\eta\ (x,y))<0,\sum_{j=1}^{m}(\mu_j+\tau_j)\alpha_1(x,y)g'(y,\eta(x,y))<0$$

By the positivity of □0 and □1 the above inequalities reduced to

$$f'(y,\eta(x,y))<0 \quad (4.11)$$

$$\sum_{j=1}^{m}\left(\mu_j+\tau_j\right)g'(y,\eta(x,y))<0 \quad (4.12)$$

But $\xi \geq 0$, (4.11) and (4.12) imply (4.8), which contradicts (4.1). This completes the proof.

Theorem 4.2: (Strong duality)
Let $\bar{x}$ be a locally weak pareto efficient solution for (P) at which the generalized slaters constraint qualification is satisfied. Let f, g be directionally differentiable at $\bar{x}$ with $f'(\bar{x},\eta\ (x,\bar{x}))$ and $g'(\bar{x},\eta\ (x,\bar{x}))$ are □-preinvex function on X .
Let g_j be continuous for $j\in\bar{J}(\bar{x})$, then there exist $\bar{\mu}\in R_+^m$ and $\bar{\tau}\in R_+^m$ such that $(\bar{x},\ 1,\bar{\mu},\bar{\tau})$ is a feasible for (MWD). If the weak duality between (P) and (MWD) in Theorem 4.1 holds, then $(\bar{x},\ 1,\bar{\mu},\bar{\tau})$ is a locally weak pareto efficient solution for (MWD).

Profit: Since $\bar{x}$ satisfies all the conditions of Lemma 2.2 there exist $\bar{\mu}\in R_+^m$ and $\bar{\tau}\in R_+^m$ such that condition (1) – (3) hold. By (1) – (3), we have $\left(\bar{x},1,\bar{\mu},\bar{\tau}\right)$ is feasible for (MWD).

By the weak duality it follows that $\left(\bar{x},1,\bar{\mu},\bar{\tau}\right)$ is a locally weak pareto efficient solution for (MWD).

Theorem 4.3: (Converse duality): Let $\left(\bar{y},\bar{\xi},\bar{\mu},\bar{\tau}\right)$ be a weak pareto efficient solution for (MWD). Moreover we assume that the hypothesis of Theorem 3.1 hold for $\bar{y}$ in $D\cup Pr_x$ W. then $\bar{y}$ is a weak pareto efficient solution for (P).
Proof: We prove the theorem by contradiction. Suppose that ($\bar{y}$ is not a weak pareto efficient solution for (P), that is, there exist $\bar{x}\in D$, such that

$f(\bar{x})<f(\bar{y})$ since condition (a) if theorem 4.1 holds, we get,

$$\sum_{i=1}^{k}\overline{\overline{\xi}_i}\,\alpha_0(\overline{x},\overline{y})\,f_i'(\overline{y},\eta(\overline{x},\overline{y}))<0$$

By the positivity of α_0 the above inequality reduces to

$$\sum_{i=1}^{k}\overline{\overline{\xi}_i}\;f_i'(\overline{y},\eta(\overline{x},\overline{y}))<0 \tag{4.13}$$

From the feasibility of $\overline{x}$ and $\left(\overline{y},\overline{\xi},\overline{\mu},\overline{\tau}\right)$ for (P) and (MWD) respectively, we have

$$\sum_{j=1}^{m}\left(\overline{\mu}_j+\overline{\tau}_j\right)g_j(\overline{y})\leq 0$$

The above inequality in the light of condition (a) of theorem 4.1, yields

$$\sum_{j=1}^{m}\left(\overline{\mu}_j+\overline{\tau}_j\right)\alpha_1(\overline{x},\overline{y})g_j'(\overline{y},\eta(\overline{x},\overline{y}))\leqq 0$$

Since $\alpha_1>0$, we get,

$$\sum_{j=1}^{m}\left(\overline{\mu}_j+\overline{\tau}_j\right)g_j'(\overline{y},\eta(\overline{x},\overline{y}))\leqq 0 \tag{4.14}$$

By (4.13) and (4.14) we get,

$$\sum_{i=1}^{k}\overline{\xi}_i\;f_i'(\overline{y},\eta(\overline{x},\overline{y}))+\sum_{i=1}^{m}\left(\overline{\mu}_j+\overline{\tau}_j\right)\;g_j'(\overline{y},\eta(\overline{x},\overline{y}))<0 \tag{4.15}$$

This contradicts the dual constraints (4.1)

Similarly by condition (b) in Theorem (4.1) we get

$$\sum_{i=1}^{k}\overline{\overline{\xi}_i}\;\alpha_0(\overline{x},\overline{y})f_i'(\overline{y},\eta(\overline{x},\overline{y}))<0,\sum_{j=1}^{m}\left(\overline{\mu}_j+\overline{\tau}_j\right)\alpha_1(\overline{x},\overline{y})\;g_j'(\overline{y},\eta(\overline{x},\overline{y}))<0$$

By the positivity of α_0 and α_1 the above two inequalities reduces to

$$\sum_{i=1}^{k}\overline{\overline{\xi}_i}\;f_t'(\overline{y},\eta(\overline{x},\overline{y}))<0,\sum_{j=1}^{m}\left(\overline{\mu}_j+\overline{\tau}_j\right)\overline{g_j}\,(\overline{y},\eta(\overline{x},\overline{y}))\leq 0$$

Since $\xi\geq 0$ the above two inequalities imply (4.15), which yields contradition (4.1). By condition (c) we have,

$$\sum_{i=1}^{k}\overline{\overline{\xi}_i}\,\alpha_0(\overline{x},\overline{y})f_i'(\overline{y},\eta(\overline{x},\overline{y}))<0,\;\;\sum_{j=1}^{m}\left(\overline{\mu}_j+\overline{\tau}_j\right)\alpha_1(\overline{x},\overline{y})\;g_j'(\overline{y},\eta\;(\overline{x},\overline{y}))<0$$

By the positivity of α_0 and α_1 the above two inequality reduces to

$$\sum_{i=1}^{k}\overline{\overline{\xi}_i}\,f_i'(\overline{y},\eta(\overline{x},\overline{y}))<0,\;\;\sum_{j=1}^{m}\left(\overline{\mu}_j+\overline{\tau}_j\right)g_j'(\overline{y},\eta\,(\overline{x},\overline{y}))<0$$

Since $\xi\geq 0$ the above two inequalities imply (4.15), which yields again a contradiction to (4.1). Hence, the proof is completed.

General Mond-Weir Duality:-

We shall continue our discussion on duality for (P) in the present section by considering the general Mond-Weir type dual problem and proving weak, and strong duality theorem under the assumption of type-I α-invex introduced in section 2.

We consider the following general Mond-Weir type dual to (P).

(GMWD) Maximize $\phi(y,\xi,\mu,\tau)=f(y)+\left(\mu_{J_0}^T+\tau_{J_0}^T\right)g_{J_0}(y)e$

subject $(\xi^T f'+(\mu^T+\tau^T))g'(y,\eta(x,y))\geq 0\,\forall x\in D$ (5.1)

$$\left(\mu_{J_t}+\tau_{J_t}\right)g_{J_t}(y)\geq 0, 1\leq t\leq r \tag{5.2}$$

$$\xi^T e = 1 \tag{5.3}$$

$$\xi\in R^k_+, \mu\in R^m_+,\ \tau\in R^m_+$$

where $J_t, 1\leq t\leq r$ are partitions of set M and e = (1, 1, ….1) $\in R^k$

Let $$W = \begin{cases} (y,\xi,\mu,\tau)\in x\times R^k\times R^m : \xi^T f'(y,\eta(x,y)+\left(\mu^T+\tau^T\right)g'(y,\eta(x,y))\geq 0 \\ \left(\mu_j+\tau_j\right)g_j(y)\geq 0,\ \ j=1,2,...m \\ \xi\in R^k_+,\ \xi^T e=1,\ \mu\in R^m_+, \tau\in R^m_+ \end{cases}$$

denote the set of all feasible solution of (GMWD)

Theorem 5.1: (Weak Duality)

Let x and $\{y,\xi,\mu,\tau\}$ be a feasible solution for (P) and (GMWD) respectively.

Assume that one of the following condition holds.

(a) $\xi>0$ and $\left(f+\left(\mu_{J_0}+\tau_{J_0}\right)g_{J_0},\left(\mu_{J_t}+\tau_{J_t}\right)g_{J_t}\right)$ strong pseudo - quasi type-I α-invex at y and $D\cup Pr_x W$ with respect to same α_0,α_1 and η for any t, $1\leq t\leq r$.

(b) $\left(f+\left(\mu_{J_0}+\tau_{J_0}\right)g_{J_0},\left(\mu_{J_t}+\tau_{J_t}\right)g_{J_t}\right)$ is weak strictly pseudo-quasi type-I, α-invex at y on $D\cup Pr_x W$ with respect to some α_0,α_1 and η for any t, $1\leq t\leq r$.

(c) $\left(f+\left(\mu_{J_0}+\tau_{J_0}\right)g_{J_0},\left(\mu_{Jt}+\tau_{J_t}\right)g_{J_t}\right)$ is weak strictly pseudo type-I α-invex at y on $D\cup Pr_x W$ with respect to some α_0,α_1 and η for any t, $1\leq t\leq r$.

Then the following condition cannot hold:

$$f(x)\leq\phi(y,\xi,\mu,\tau)$$

Proof: We prove the theorem by contradiction

suppose,

$$f(x)\leq\phi(y,\xi,\mu,\tau) \tag{5.4}$$

Since x is feasible for (P) and $\mu\geq 0,\tau\geq 0$, (5.4) implies that

$$f(x)+\left(\mu^T_{J_0}+\tau^T_{J_0}\right)g_{J_0}(x)e\leq f(y)+\left(\mu^T_{J_0}+\tau^T_{J_0}\right)g_{J_0}(y)e$$

$$\Rightarrow f(x)+\left(\mu^T_{J_0}+\tau^T_{J_0}\right)g_{J_0}(x)e-f(y)-\left(\mu^T_{J_0}+\tau^T_{J_0}\right)g_{J_0}(y)e\leq 0 \tag{5.5}$$

From the feasible of x for (P) and (5.2), we have,

$$-\left(\mu^T_{J_t}+\tau^T_{J_t}\right)g_{J_t}\leqq 0 \quad \text{for any t, } 1\leq t\leq r \tag{5.6}$$

By condition (a) from (5.5) and (5.6) we have,

$$\alpha_0(x,y)f'(y,\eta(x,y))+\left(\mu_{J_0}+\tau_{J_0}\right)\alpha_0(x,y)g'_{J0}(y,\eta(x,y))\leq 0$$

and $\left(\mu_{J_t}+\tau_{J_t}\right)\alpha_1(x,y)g'_{J_t}(y,\eta(x,y))\leq 0,$ for any $1\leq t\leq r$

By the positivity of α_0 and α_1 the above two inequalities reduce to.

$$f'(y,n(x,y)+\left(\mu_{J_0}+\tau_{J_0}\right)g'_{J0}(y,\eta(x,y))\leq 0$$

and $\left(\mu_{J_t}+\tau_{J_t}\right)g'_{J_t}(y,\eta(x,y))\leq 0,$ for any t $1\leq t\leq r$

Since, $\xi > 0$ the above two inequalities yield.

$$f'(y,\eta(x,y)) + \sum_{t=0}^{r}\left(\mu_{J_t} + \tau_{J_t}\right) g'_{J_t}(y,\eta(x,y)) < 0 \quad (5.7)$$

Since $J_0, J_1, J_2, \ldots, J_r$ are partition of M (5, 7) is equivalent to

$$f'(y,\eta(x,y)) + \left(\mu^T + \tau^T\right) g'(y,\eta(x,y)) < 0 \quad (5.8)$$

which contradicts to dual constraint (5.2).
Similarly by condition (b) we have

$$\alpha_0(x,y) f'(y,\eta(x,y)) + \left(\mu_{J_0} + \tau_{J_0}\right)\alpha_0(x,y) g'_{J_0}(y,\eta(x,y)) < 0$$

and $(\mu_{J_t} + \tau_{J_t})\alpha_1(x,y) g'_{J_t}(y,\eta(x,y)) \le 0$, for any t $1 \le t \le r$

By the positivity of α_0 and α_1 the above two inequalities reduces to,

$$f'(y,\eta(x,y)) + \left(\mu_{J_0} + \tau_{J_0}\right) g'_{J_0}(y,\eta(x,y)) < 0$$

and $\left(\mu_{J_t} + \tau_{J_t}\right) g'_{J_t}(y,\eta(x,y)) \le 0$ for any $1 \le t \le r$,

Since $\xi \ge 0$, the above two inequalities yield,

$$f'(y,\eta(x,y)) + \sum_{t=0}^{r}\left(\mu_{J_t} + \tau_{J_t}\right) g'_{J_t}(y,\eta(x,y)) < 0$$

The above inequality leads to (5.8) which contradicts (5.1)
Now for the part (c) following the similar process, we get (5.8), which contradicts (5.1)
Hence the proof is completed.

Theorem (5.2) (Strong duality)
Let $\bar{x}$ be a locally weak pareto efficient solution for (P) at which the generalized slaters constraint qualification is satisfied. Let f, g be directionally differentiable at $\bar{x}$ with $f'(\bar{x},\eta(x,\bar{x}))$ and $g'(\bar{x},\eta(x,\bar{x}))$ are α-preinvex functions on X.

Let g_j be continuous for $j \in \bar{J}(\bar{x})$, then theorem exist $\bar{\mu} \in R^m_+, \bar{\tau} \in R^m_+$ such that $(\bar{x}, 1, \bar{\mu}, \bar{\tau})$ is feasible for (GMWD). If the weak duality between (P) and (MWD) in Theorem 5.1 holds, then $(\bar{x}, 1, \bar{\mu}, \bar{\tau})$ is a locally weak pareto efficient solution for (GMWD).

Proof: The proof of this theorem is similar to the proof of Theorem 4.2.

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