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RESEARCH ARTICLE

NUMERICAL SOLUTION OF REYNOLDS EQUATION USING DIFFERENTIAL TRANSFORM METHOD.

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Abstract

Reynolds equation is a partial differential equation, derived from the Navier-Stokes equations. Reynolds equation is the fundamental equations of the hydrodynamic lubrication theory. Solution of Reynolds equation describes the pressure distribution of the lubricant in a journal bearing with finite length. The parameters involved in the Reynolds equation are viscosity, density and film thickness of lubricant. However, an accurate analysis of the fluid film hydrodynamics obtained using many numerical solution of the Reynolds equation. Differential Transform Method (DTM) is one of the powerful numerical methods applied to solve linear and nonlinear partial differential equations. This study aims to apply DTM to solve Reynolds equation in partial differential form to get pressure distribution of journal bearing. Results obtained from the DTM compared with available solutions obtained using other numerical methods and show good agreement. The obtained results reveal that the technique used here is good, effective and convenient for such kind of problems.

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Introduction:-

Most phenomena in real world are described through nonlinear equations. Nonlinear phenomena play important roles in applied mathematics, physics and in engineering problems in which each parameter varies depending on different factors. The importance of obtaining the exact or approximate solutions of nonlinear partial differential equations (NLPDEs) in physics and mathematics, it is still a hot spot to seek new methods to obtain new exact or approximate solutions. Large class of nonlinear equations does not have a precise analytic solution, so numerical methods have largely been used to handle these equations.

The concept of differential transform method was first introduced by Zhou [1] in 1986 and it was used to solve both linear and nonlinear initial value problems in electric circuit analysis. The main advantage of this method is that it can be applied directly to NLPDEs without requiring linearization, discretization, or perturbation. It is a semi analytical-numerical technique that formulates Taylor series in a very different manner. This method constructs, for differential equations, an analytical solution in the form of a polynomial. Chen and Ho [2] solved Partial Differential Equations (PDE) is proposed in this study by using two-dimensional differential transform. First, the theory of two-dimensional differential transform is introduced. Second, taking two-dimensional differential transform of a PDE problem, a set of difference equations is derived. Doing some simple mathematical operations on these equations.

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Finally, three PDE problems with constant and variable coefficients are solved by the present method. The calculated results are compared very well with those obtained by other analytical or approximate methods.

Jang et al. [3] present the definition and operation of the two-dimensional differential transform. A distinctive feature of the differential transform is its ability to solve linear and nonlinear differential equations. Partial differential equation of parabolic, hyperbolic, elliptic and nonlinear types can be solved by the differential transform. Hassan [4] studied the differential transformation technique which is applied to solve eigenvalue problems and to solve partial differential equations. First, using the one-dimensional differential transformation to construct the eigenvalues and the normalized eigenfunctions for the differential equation of the second- and the fourth-order. Second, using the two-dimensional differential transformation to solve P.D.E. of the first- and second-order with constant coefficients. In both cases, a set of difference equations is derived and the calculated results are compared closely with the results obtained by other analytical methods.

Ayaz [5] studied two-dimensional differential transform method of solution of the initial value problem for partial differential equations. New theorems have been added and some linear and nonlinear PDEs solved by using this method. The method can be easily applied to linear or nonlinear problems and is capable of reducing the size of computational work. Ayaz [6] introduced three-dimensional differential transform method and fundamental theorems have been defined for the first time. Moreover, as an application of two and three-dimensional differential transform, exact solutions of linear and non-linear systems of partial differential equations have been investigated. The results of the present method are compared very well with those obtained by decomposition method. Differential transform method can easily be applied to linear or non-linear problems and reduces the size of computational work.

Kurnaz et al. [7] solved partial differential equations (PDEs) using the generalization of the differential transformation method to n-dimensional case. A distinctive practical feature of this method is its ability to solve especially nonlinear differential equations efficiently. The results applied to a few initial boundary-value problems to illustrate the proposed method. Hassan [8] compared the differential transformation method DTM and Adomian decomposition method ADM to solve partial differential equations (PDEs). A distinctive practical feature of the differential transformation method DTM is its ability to solve linear or nonlinear differential equations. Higher-order dimensional differential transformations are applied to a few some initial value problems to show that the solutions obtained by the proposed method DTM coincide with the approximate solution ADM and the analytic solutions.

Murat DUZ and Ugur ILTER. [9] give differential transforms of first, second and third derivatives of a complex function. Later, third order complex equations were solved using two dimensional differential transform. Kangalçil and Ayaz [10] present a reliable algorithm in order to obtain exact and approximate solutions for the nonlinear dispersive KdV and mKdV equations with initial profile. The approach rests mainly on two-dimensional differential transform method which is one of the approximate methods. The method can easily be applied to many linear and nonlinear problems and is capable of reducing the size of computational work. Exact solutions can also be achieved by the known forms of the series solutions.

Elrod-Adams model [11-12] is not straightforwardly accomplished with the FEM formulation. Essentially, the main difficulties arise in the discretization of the convective term of the modified equation, as well as in the enforcement of the flow conservation on the cavitation boundaries throughout the lubricated contact.

Elrod and Brewster [13] developed a numerical reduction approach to solve the Reynolds equation coupled with the 2D energy equation with Dirichlet boundary conditions. Temperature and fluidity (inverse of viscosity) are approximated by third order Legendre polynomials across the fluid film thickness. Elrod used Lobatto point quadrature method to discretize and calculate the integral quantities across the film thickness. The pressure and temperature are discretized using the classical finite difference methods in the other directions. The method showed good agreement with classical approaches.

Elrod [14] improved the precision of the method by approximating the temperature and the fluidity using arbitrary orders Legendre polynomials. In 2005, Moraru [15] extends the approach presented by Elrod [14] to compressible fluids and takes also into account a temperature-dependent density. In his work, a 2D formulation of the energy equation neglecting the axial heat conduction is used. In contrast to [13] and [14], the density is also approximated

by Legendre polynomials across the fluid film thickness. The governing partial differential equations are solved by finite difference methods with upwind scheme for numerical stability.

In 2009, Feng and Kaneko [16] used the same approach as Moraru to calculate the temperature and the pressure distributions in a multi-wound foil bearing while taking into account foil deflections. Unlike Moraru, Feng and Kaneko solved the energy equation on a 3D computational domain using finite difference methods. In 2015, Mahner et al. [17] used the reduction approach to analyze steady state performances of thrust and slider bearings operating with a compressible fluid. The authors used the Quadrature Method, the Modified Quadrature Method, Lobatto Point Collocation Method and the Galerkin Method in order to reduce number of unknowns of the discretized equations. According to the authors, all these methods yielded a significant time reduction compared to the classical methods.

Silun Zhang et al. [18] present numerical solution of the Reynolds equation coupled with the energy transport equation. A Spectral approach named Lobatto Point Collocation Method (LPCM) is studied. The combination of LPCM with two different film rupture/reformation models is validated using numerical results published in the literature in the cases of 1D slider. Sfyris and Chasalevris [19] solve the Reynolds equation for the pressure distribution of the lubricant in a journal bearing with finite length analytically. Using the method of separation of variables and compare the results with past numerical solutions.

Nomenclature:

P resulting pressure of the lubricant	φ_0 attitude angle of the journal
x axial coordinate of the bearing	C_r bearing radial clearance
θ angular coordinate of the bearing	e journal eccentricity
h fluid film thickness	\dot{e} journal eccentricity rate of change
R journal radius	ε journal eccentricity ratio
Ω journal rotational speed	$\dot{\varepsilon}$ journal eccentricity ratio rate of change
μ lubricant dynamic viscosity	L_b bearing/journal length

Mathematical modeling:

The problem of the lubrication of journal bearings with finite length is defined in this work as the calculation of the pressure distribution of the Newtonian lubricant that is assumed to flow under laminar, isoviscous, and isothermal conditions in between the rotating journal and the static bearing. The journal of radius R and length L_b is assumed to be rotating with a constant rotational speed and to be constantly located in a point of eccentricity e with respect to the geometric center of the bearing of radius $R+C_r$ and length L_b after an application of a virtual vertical load W as shown in Fig. 1.

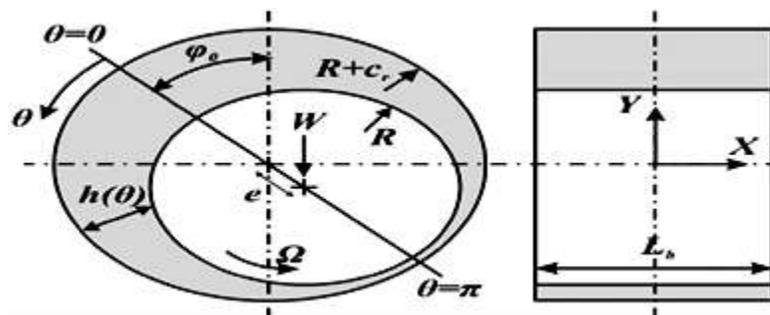


Fig.1:-De finition of the coordinate system and of the parameters of operation and design in a plain cylindrical journal bearing.

The load is not used as a parameter in this work since no forces are evaluated or expressed and the unique “inputs” in the pressure evaluation are considered to be the eccentricity e and its rate of change \dot{e} . The journal and the bearing are supposed to be in parallel (aligned bearing) and the fluid film thickness h becomes a function of the

unique independent parameter θ for a time moment of constant e and \dot{e} which means that the function for the fluid film thickness is $h = C_r + e \cos(\theta)$ and its time derivative is $\partial h / \partial t = \dot{e} \cos(\theta)$. The dynamic viscosity of the lubricant is assumed to be constant and equal to μ through the entire control volume (notified with shadow in Fig. 1) that is defined from the bearing and the journal surfaces. The attitude angle of the journal is defined as φ_0 with respect to the vertical coordinate axis (see Fig. 1). The starting point is the equation of Reynolds which is expressed as

$$\frac{\partial}{\partial x} \left(\frac{h^3}{6\mu} \frac{\partial P(x, \theta)}{\partial x} \right) + \frac{1}{R^2} \frac{\partial}{\partial \theta} \left(\frac{h^3}{6\mu} \frac{\partial P(x, \theta)}{\partial \theta} \right) = \Omega \frac{\partial h}{\partial \theta} + 2 \frac{\partial h}{\partial t} \quad (2.1)$$

After substituting the fluid film thickness function of Eq. (2.2) into Eq. (2.1) and performing the derivations one will arrive at Eq. (2.3):

$$h = C_r + e \cos(\theta) \quad (2.2)$$

$$\frac{(C_r + e \cos(\theta))^3}{6\mu} \frac{\partial^2 P(x, \theta)}{\partial x^2} - \frac{3(C_r + e \cos(\theta))^2 e \sin(\theta)}{6\mu R^2} \frac{\partial P(x, \theta)}{\partial \theta} + \frac{(C_r + e \cos(\theta))^3}{6\mu R^2} \frac{\partial^2 P(x, \theta)}{\partial \theta^2} = -e \Omega \sin(\theta) + 2 \dot{e} \cos(\theta) \quad (2.3)$$

The Boundary conditions are

$$p(x, 0) = p(x, 2\pi) = p(0, \theta) = p(L, \theta) = 0 \text{ and } \frac{\partial p}{\partial x} = 0 \text{ at the outlet}$$

Eq. (2.3) is the one that we are going to work with.

Basic ideas of the differential transform method

The basic definitions and fundamental operations of the two-dimensional differential transform are defined in [2-9]. Consider a function of two variables $p(x, \theta)$ be analytic in the domain Ω and let $(x, \theta) = (x_0, \theta_0)$ in this domain. The function $p(x, \theta)$ is then represented by one series whose center at located at $p(x_0, \theta_0)$. The differential transform of the function is the form

$$P(k, h) = \frac{1}{k!h!} \left[\frac{\partial^{k+h} p(x, \theta)}{\partial x^k \partial \theta^h} \right]_{(x_0, \theta_0)} \quad (3.1)$$

where $p(x, \theta)$ is the original function and $P(k, h)$ is the transformed function. The differential inverse transform of $P(k, h)$ is defined as

$$p(x, \theta) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} P(k, h) (x - x_0)^k (\theta - \theta_0)^h \quad (3.2)$$

The relations (3.1) and (3.2) imply that

$$p(x, \theta) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k!h!} \left[\frac{\partial^{k+h} p(x, \theta)}{\partial x^k \partial \theta^h} \right]_{(x_0, \theta_0)} x^k \theta^h (x - x_0)^k (\theta - \theta_0)^h \quad (3.3)$$

In a real application, and when (x_0, θ_0) are taken as $(0, 0)$, then the function $p(x, \theta)$ is expressed by a finite series and Eq. (3.2) can be written as

$$p(x, \theta) \cong \sum_{k=0}^m \sum_{h=0}^n [P(x, \theta)] x^k \theta^h \quad (3.4)$$

in addition Eq. (3.4) implies that $\sum_{k=m+1}^{\infty} \sum_{h=n+1}^{\infty} [P(x, \theta)] x^k \theta^h$

is negligibly small. Usually, the values of m and n are decided by convergences of the series coefficients. The fundamental mathematical operations performed by two-dimensional differential transform method are listed in Table 1.

Table 1:-The operations for the two-dimensional differential transform method.

Original function	Transformed function
$p(x, \theta) = u(x, \theta) \pm v(x, \theta)$	$P(k, h) = U(k, h) \pm V(k, h)$
$p(x, \theta) = \alpha u(x, \theta)$	$P(k, h) = \alpha U(k, h), \alpha$ is constant
$p(x, \theta) = \frac{\partial u(x, \theta)}{\partial x}$	$P(k, h) = (k + 1)U(k + 1, h)$
$p(x, \theta) = \frac{\partial u(x, \theta)}{\partial \theta}$	$P(k, h) = (h + 1)U(k, h + 1)$
$p(x, \theta) = u(x, \theta)v(x, \theta)$	$P(k, h) = \sum_{r=0}^k \sum_{s=0}^h U(r, h - s)V(k - r, s)$
$p(x, \theta) = x^m \theta^n$	$P(k, h) = \delta(k - m, h - n) = \delta(k - m)\delta(h - n)$ where $\delta(k - m) = \begin{cases} 1, & k = m, \\ 0, & k \neq m, \end{cases} \delta(h - n) = \begin{cases} 1, & h = n \\ 0, & h \neq n \end{cases}$
$p(x, \theta) = \frac{\partial^{r+s} u(x, \theta)}{\partial x^r \partial \theta^s}$	$P(k, h) = (k + 1)(k + 2) \dots (k + r)(h + 1)(h + 2) \dots (h + s)U(k + r, h + s)$
$p(x, \theta) = \sin \theta$	$P(k, h) = \frac{1}{h!} \sin\left(\frac{\pi h}{2}\right)$
$p(x, \theta) = \cos \theta$	$P(k, h) = \frac{1}{h!} \cos\left(\frac{\pi h}{2}\right)$

Mathematical solution:

Taking the two-dimensional transform of Eq. (2.3) by using the related definitions in Table 1, we have

$$\begin{aligned}
 & R^2 \left(C_r^3 + \frac{3}{2} C_r e^2 \right) + \left[3C_r^2 + \frac{3}{4} e^3 + \frac{3}{2} C_r e^2 2^h + \frac{e^3}{4} 3^h \right] \frac{R^2}{h!} \cos\left(\frac{\pi h}{2}\right) (k + 1)(k + 2)U(k + 2, h) \\
 & - \left[(3C_r^2 e + \frac{3}{2} e^2) \frac{1}{h!} \sin\left(\frac{\pi h}{2}\right) + 6C_r e^2 \frac{1}{h! h!} \cos\left(\frac{\pi h}{2}\right) \sin\left(\frac{\pi h}{2}\right) + \frac{3 e^3 2^h}{2 h! h!} \cos\left(\frac{\pi h}{2}\right) \sin\left(\frac{\pi h}{2}\right) \right] (h + 1) \\
 & \times U(k, h + 1) + \left[C_r^3 + \frac{3}{2} C_r e^2 + 3C_r^2 + \frac{3}{4} e^3 + \frac{3}{2} C_r e^2 2^h + \frac{e^3}{4} 3^h \right] \frac{1}{h!} \cos\left(\frac{\pi h}{2}\right) (h + 1)(h + 2)U(k, h + 2) \\
 & = -6\mu R^2 e \Omega \frac{1}{h!} \sin\left(\frac{\pi h}{2}\right) + 12\mu R^2 \dot{e} \Omega \frac{1}{h!} \cos\left(\frac{\pi h}{2}\right) \tag{4.1}
 \end{aligned}$$

By applying the two-dimensional transform of boundary conditions by using the related definitions in Table 1, we have $U(k, 0) = U(k, 2\pi) = U(0, h) = U(L, h) = 0, U(k, 1) = 0$ (4.2) then

substitute Eq.(4.2) in Eq.(4.1) with $\varepsilon = e / c_r, \frac{e^\square}{\Omega R} = 0.001$ and by recursive method we can calculate another values of $U(k, h)$.

$$p(x, \theta) \cong \sum_{k=0}^m \sum_{h=0}^n [U(h, k)] x^k \theta^h$$

$$= U(0,0) + U(1,0)x + U(0,1)\theta + U(1,1)x\theta + \dots + U(m,n)x^m\theta^n$$

Results:-

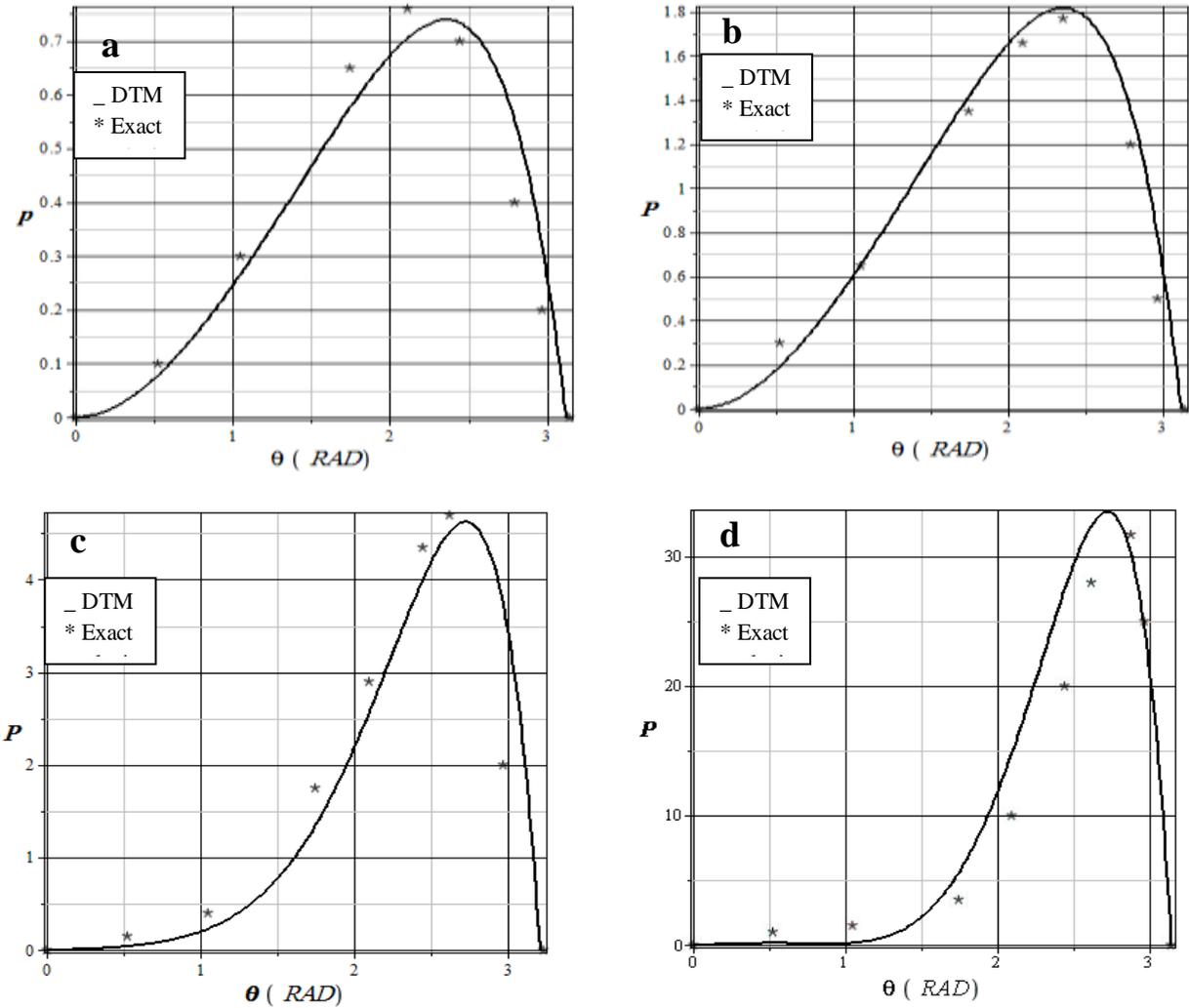


Fig.2:-the resulting pressure $p(x, \theta)$ along angular coordinate θ in the axial center of bearing ($x=0$) for variable values of eccentricity ratio. (a) $\epsilon=0.3$, (b) $\epsilon=0.5$, (c) $\epsilon=0.7$, (d) $\epsilon=0.9$ and $L/D=1$ in all diagrams.

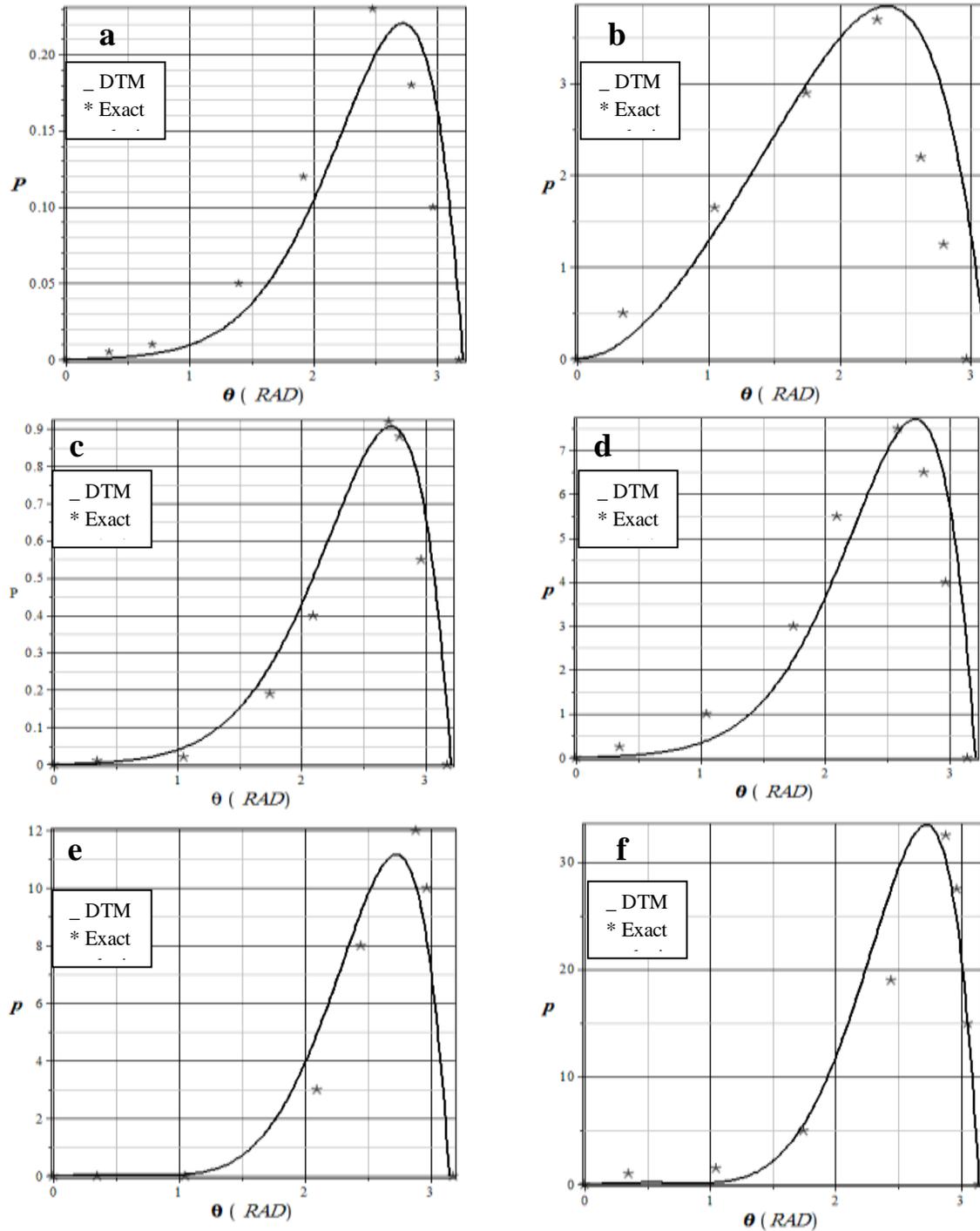


Fig.3:the resulting pressure $p(x, \theta)$ along angular coordinate θ in the axial center of bearing ($x=0$) for variable values of eccentricity ratio. (a) $\epsilon=0.5$ and $L/D=0.25$, (b) $\epsilon=0.5$ and $L/D=4$, (c) $\epsilon=0.7$ and $L/D=0.25$, (d) $\epsilon=0.7$ and $L/D=4$, (e) $\epsilon=0.9$ and $L/D=0.25$, (f) $\epsilon=0.9$ and $L/D=4$

Results of the proposed numerical technique applied to Reynolds equation of lubrication are shown in Figs. 2 and 3. The DTM numerical method results presented show good agreement with exact solution in [19]. There are some differences in the maximum values of pressure but also in the domain of maximum values. Comparing now the DTM results with the exact analytical result. In (Fig. 2a) we can see in exact analytic solution that the maximum value of

$P=0.765$ at angle $\theta=2.15$ (rad) and in DTM method the maximum value of $P=0.745$ at angle $\theta=2.35$ (rad). In (Fig. 2b) we can see in exact analytic solution that the maximum value of $P=1.79$ at angle $\theta=2.3$ (rad) and in DTM method the maximum value of $P=1.82$ at angle $\theta=2.3$ (rad). In (Fig. 2c) we can see in exact analytic solution that the maximum value of $P=4.73$ at angle $\theta=2.60$ (rad) and in DTM method the maximum value of $P=4.62$ at angle $\theta=2.70$ (rad). In (Fig. 2d) we can see in exact analytic solution that the maximum value of $P=32$ at angle $\theta=2.80$ (rad) and in DTM method the maximum value of $P=34$ at angle $\theta=2.75$ (rad). In (Fig. 3a) we can see in exact analytic solution that the maximum value of $P=0.23$ at angle $\theta=2.50$ (rad) and in DTM method the maximum value of $P=0.22$ at angle $\theta=2.65$ (rad). In (Fig. 3b) we can see in exact analytic solution that the maximum value of $P=3.65$ at angle $\theta=2.250$ (rad) and in DTM method the maximum value of $P=3.8$ at angle $\theta=2.35$ (rad). In (Fig. 3c) we can see in exact analytic solution that the maximum value of $P=0.92$ at angle $\theta=2.750$ (rad) and in DTM method the maximum value of $P=0.91$ at angle $\theta=2.750$ (rad). In (Fig. 3d) we can see in exact analytic solution that the maximum value of $P=7.5$ at angle $\theta=2.60$ (rad) and in DTM method the maximum value of $P=7.7$ at angle $\theta=2.70$ (rad). In (Fig. 3e) we can see in exact analytic solution that the maximum value of $P=12$ at angle $\theta=2.80$ (rad) and in DTM method the maximum value of $P=11.1$ at angle $\theta=2.75$ (rad). In (Fig. 3e) we can see in exact analytic solution that the maximum value of $P=32.5$ at angle $\theta=2.850$ (rad) and in DTM method the maximum value of $P=43$ at angle $\theta=2.78$ (rad) which converges to the exact solution.

The differences of the DTM solution in comparison to the exact pressure solution are minimal for all of these four cases of eccentricity ratio, which correspond to a heavily loaded bearing, to a lightly loaded bearing and also to intermediate cases. The DTM solution seems to slightly under-estimate the exact pressure in the cases of $\varepsilon=0.3$ (Fig. 2a), $\varepsilon=0.7$ (Fig. 2c), $\varepsilon=0.5$ (Fig. 3a) and $\varepsilon=0.7$ (Fig. 3d), while in the case of $\varepsilon=0.5$ (Fig. 2b), $\varepsilon=0.9$ (Fig. 2d), $\varepsilon=0.5$ (Fig. 3b) and $\varepsilon=0.9$ (Fig. 3f) the DTM solution seems to overestimate the exact pressure.

In Fig. 3 we present two cases of L/D ratio corresponding to a short ($L/D=0.25$) and a long ($L/D=4$) bearing with evaluating the pressure profile under the use of the exact analytical solution, the FDM, and the Short/Long bearing approximation for three different cases of $\varepsilon=0.5$, $\varepsilon=0.7$ and $\varepsilon=0.9$. As shown in Fig. 3b, d and f, the long bearing approximation yields results of almost absolute agreement with the exact analytical solution and with slight differences to the numerical results. The short bearing approximation, see Fig. 3a, c and e, appeared with a very good agreement in cases of eccentricity ratio $\varepsilon=0.5$ and $\varepsilon=0.7$, and only slight differences are noticed in the maximum pressure of the heavily loaded short bearing in Fig. 3e. However, in all cases these slight differences presented between the pressure distributions could be explained as a matter of the differences in the nature of the solutions (DTM, analytical exact).

Conclusion:-

In this work, we have successfully developed DTM to obtain an approximation to the solution of the Reynolds equation. It is apparent that this method is a very influential and efficient technique. There is no need for linearization or perturbations; large computational work and round-off errors are avoided. The results obtained demonstrate the reliability of the algorithm and its applicability to some partial differential equations. It provides more realistic series solutions that converge very rapidly in real physical problems. It may be also concluded that DTM is very powerful and reliable in finding analytical as well as numerical solutions for wide classes of nonlinear differential equations.

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