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### RESEARCH ARTICLE

#### PERSPECTIVES ON BALKING AND RENEGING IN M/M/1/K QUEUES

Pallabi Medhi and Amit Choudhury

Dept. of Statistics, Gauhati University, Guwahati-781014, Assam, India.

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#### Abstract

The analysis of a single server finite buffer Markovian queuing system with state-independent balking and Markovian reneging has been presented in this paper. Reneging till beginning and reneging till end of service have been considered separately. Closed form expressions of number of performance measures have been presented. Sensitivity analysis has also been performed to examine the variation in performance measures with the variation in system parameters. A numerical example has been presented to demonstrate results derived.

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#### Introduction and Motivation:

It has been close to a century that the theory of queues has been developing. In the process, many queuing scenarios have been modeled. Many of such scenarios have been motivated from real life experiences. Of the various characteristics of customer behavior in a typical queuing system, this paper deals with the phenomenon of balking and reneging. These days, customers are very hard pressed for time. Additionally, they desire prompt and efficient service. It is our day-to-day experience where a customer on its arrival to the queuing system decides against joining the queue if there is one already in existence. The act of not joining the queue by the customer and leaving the queuing system is known as balking. Customers may balk due to various reasons. Haight (1957) has provided a rationale, which might influence a person to balk. It relates to the perception of the importance of being served which induces an opinion somewhere in between urgency so that a queue of certain length will not be joined to indifference where a non-zero queue is also joined. Even if a customer does not balk and joins the system, it is very often the case where customers are unwilling to wait as long as it is necessary to complete the act of receiving service. In other words, customers may not be indefinitely patient. Consequently a customer who has joined the system may depart either from the queue or from the service station without completely receiving the service. This is known as reneging. Even though reneging is a common phenomenon in any queuing system, yet it is not very often that one can locate a paper on reneging.

Reneging can be of two types- reneging till beginning of service (henceforth referred to as R\_BOS) and reneging till end of service (henceforth referred to as R\_EOS). A customer can renege only as long as it is in the queue and this can be observed as R\_BOS. It cannot renege once it begins receiving service. A common example is the barbershop. A customer can renege while he is waiting in queue. However once service get starts i.e. hair cut begins, the customer cannot leave till service is over. On the other hand customer can renege not only while waiting in queue but also while receiving service, which can be observed as R\_EOS. Such a situation occurs for example in the processing or merchandising of perishable goods.

In this paper we shall analyze the Markovian single server finite buffer queue where an arriving customer may balk as well renege. A few non-traditional performance measures will also be presented. Basically the aim behind

**Corresponding Author:- Pallabi Medhi**

Address:- Dept. of Statistics, Gauhati University, Guwahati-781014, Assam, India.

constructing the additional performance measures is to reflect upon the implications of renegeing and balking in the system. Implications of change in system parameters with regard to server utilization will also be discussed.

An early work on renegeing was by Barrer (1957) where he considered deterministic renegeing with single server markovian arrival and service rates. Customers were selected randomly for service. In his subsequent work, Barrer (1958) also considered deterministic renegeing (of both R\_BOS and R\_EOS) in a multiserver scenario with FCFS discipline. Another early work was by Haight (1959). Ghosal (1963) considered a D/G/1 model with deterministic renegeing. Gavish and Schweitzer (1977) also considered a deterministic renegeing model with the additional assumption that arrivals can be labeled by their service requirement before joining the queue and arriving customers are admitted only if their waiting plus service time do not exceed some fixed amount. This assumption is met in communication systems. Choudhury (2008) analyzed a single server markovian queuing system with the added complexity of customers who are prone to giving up whenever its waiting time is larger than a random threshold-his patience time. He assumed that these individual patience times were independent and identically distributed exponential random variables. Renegeing till beginning of service was considered. A detailed and lucid derivation of the distribution of virtual waiting time in the system was presented. Some performance measures were also presented. Few other attempts at modeling renegeing phenomenon include those by Baccelli et al (1984), Kok and Tijms (1985), Martin and Artalejo (1995), Boots and Tijms (1999), Bae et al (2001), Choi et al (2001), Choi Kim and Zhu (2004), Zhang et al (2005), Singh et al (2007), Altman and Yechiali (2008), Kim et al (2008) and Liu and Kulkarni (2008).

Similarly, an early work on balking was by Haight (1957). Haghghi et al (1986) considered a Markovian multiserver queuing model with balking as well as renegeing. Each customer had a balking probability which was independent of the state of the system. R\_BOS was considered as renegeing discipline. Liu et al (1987) considered an infinite server Markovian queuing system with renegeing of type R\_BOS. Customers had a choice of individual service or batch service: batch service being preferred by the customer. Brandt et al (1998) considered a S-server system with two FCFS queues, where the arrival rates at the queues and the service may depend on number of customers 'n' being in service or in the first queue, but the service rate was assumed to be constant for  $n > s$ . The customers in the first queue were assumed impatient customers with deterministic renegeing. Wang et al (1999) considered the machine repair problem in which failed machines balk with probability (1-b) and renege according to a negative exponential distribution. Another work using the concepts of balking and renegeing in machine interference queue has been carried out by Al-Seedy and Al-Ibraheem (2001).

There have been a few papers in which both balking as well as renegeing was considered. Here mention may be made of the papers by Haghghi et al (1986), Zhang et al (2005), El-Paoumy (2008), El-Sherbiny (2008), Shawky and El-Paoumy (2009).

In this paper we discuss the analysis of M/M/1/k model with the additional restriction that a customer may balk as well as renege. The two types of renegeing R\_BOS and R\_EOS are discussed separately. To the best of our knowledge, an analysis with both types of renegeing (of Markovian type) in addition to state independent balking has not been carried out in this model. Even though one can find queuing models of various types analyzed in queuing literature, it is not often that renegeing and balking have been analyzed together. Even if these have been dealt with, explicit closed expressions are not available. This paper is an attempt in this direction. Importance of the queuing model stems from the fact that in the classical M/M/1 model," it is assumed that the system can accommodate any number of units. In practice, this may seldom be the case. We have thus to consider the situation such that the system has limited waiting space and can hold a maximum number of k units (including the one being served)" {Medhi (1994)}. As for balking, we assume that each customer has a state independent balking probability. It will be assumed that if the customer on arrival observes the existence of a queue but with system size below finite buffer restriction, the probability that he will balk is 'p'. It may be noted that our formulation requires that balking is possible only when the system is non-empty. There is no balking from an empty system. We further assume that each customer joining the system has a random patience time following  $\exp(\nu)$  distribution. This patience time commences from the time it joins the systems. In case the renegeing distribution is R\_BOS, the customer will renege i.e. leave the system in case service does not begin before expiry of this patience time. In case of R\_EOS, the customer would renege in case service is not over before the expiry of the patience time. Thus in case of R\_EOS, the customer may depart either from the queue or from the service station with partial and incomplete service whereas in case of R\_BOS, the customer can renege only from the queue. The arrival and service patterns are assumed to be Markovian with rates  $\lambda$  and  $\mu$  respectively.

The rest of the paper is structured as follows. In section 2, we obtain the system state probabilities. In section 3, performance measures are presented. Sensitivity analysis is carried out in section 4. A numerical example is discussed in section 5. Section 6 concludes the paper. Some derivations are given in the appendix which is placed in section 7.

**The System State Probabilities:**

In this section, the steady state probabilities are derived. Under R\_BOS, let  $p_n$  denote the probability that there are 'n' customers in the system. Applying the Markov process theory, we obtain the following set of steady state equations.

$$\lambda p_0 = \mu p_1 \tag{2.1}$$

$$\lambda p_0 + (\mu + \nu)p_2 = \lambda(1 - p)p_1 + \mu p_1 \tag{2.2}$$

$$\lambda(1 - p)p_{n-1} + (\mu + n\nu)p_{n+1} = \lambda(1 - p)p_n + \{\mu + (n - 1)\nu\}p_n \quad : 2 \leq n \leq k - 1, \tag{2.3}$$

$$\lambda(1 - p)p_{k-1} = \{\mu + (k - 1)\nu\}p_k .$$

Solving recursively, we get (under R\_BOS)

$$p_n = \frac{\lambda^n (1 - p)^{n-1}}{\prod_{r=1}^n \{\mu + (r - 1)\nu\}} p_0 \quad : n=1, 2, \dots, k, \tag{2.4}$$

where  $p_0$  is obtained from the normalizing condition  $\sum_{n=0}^k p_n = 1$  and is given as

$$p_0 = \left[ 1 + \sum_{n=1}^k \frac{\lambda^n (1 - p)^{n-1}}{\prod_{r=1}^n \{\mu + (r - 1)\nu\}} \right]^{-1} . \tag{2.5}$$

Under R\_EOS, let  $q_n$  denote the probability that there are n customers in the system. Proceeding similarly, we obtain the following set of steady state equations.

$$\lambda q_0 = (\mu + \nu)q_1, \tag{2.6}$$

$$\lambda q_0 + (\mu + 2\nu)q_2 = \lambda(1 - p)q_1 + (\mu + \nu)q_1, \tag{2.7}$$

$$\lambda(1 - p)q_{n-1} + \{\mu + (n + 1)\nu\}q_{n+1} = \lambda(1 - p)q_n + (\mu + n\nu)q_n \quad : 2 \leq n \leq k - 1, \tag{2.8}$$

$$\lambda(1 - p)q_{k-1} = (\mu + k\nu)q_k .$$

Solving, we get (under R\_EOS)

$$q_n = \frac{\lambda^n (1 - p)^{n-1}}{\prod_{r=1}^n (\mu + r\nu)} q_0 \quad : n = 1, 2, \dots, k, \tag{2.9}$$

where

$$q_0 = \left[ 1 + \sum_{n=1}^k \frac{\lambda^n (1 - p)^{n-1}}{\prod_{r=1}^n (\mu + r\nu)} \right]^{-1} . \tag{2.10}$$

**Performance Measures:**

An important measure is ‘L’, which denotes the mean number of customers in the system. To obtain an expression for the same, we note that  $L=P'(1)$  where

$$P'(1) = \frac{d}{ds} P(s) |_{s=1}.$$

Here P(S) is the p.g.f. of the steady state probabilities. The derivation of P'(1) is given in the appendix. From (7.1.1) and (7.2.1), the mean system size under two reneging rules are

$$L_{(R\_BOS)} = \frac{\lambda - (\mu - \nu + \lambda p)(1 - p_0) - \lambda(1 - p)p_k}{\nu},$$

$$L_{(R\_EOS)} = \frac{\lambda - (\mu + \lambda p)(1 - q_0) - \lambda(1 - p)q_k}{\nu}.$$

Mean queue size can now be obtained and are given by

$$L_{q(R\_BOS)} = \frac{\lambda - (\mu + \lambda p)(1 - p_0) - \lambda(1 - p)p_k}{\nu},$$

Similarly,

$$L_{q(R\_EOS)} = \frac{\lambda - (\mu - \nu + \lambda p)(1 - q_0) - \lambda(1 - p)q_k}{\nu}.$$

Customers arrive into the system at the rate of  $\lambda$ . However all the customers who arrive do not join the system either because of balking or because of finite buffer restriction. The effective arrival rate into the system is thus different from the overall arrival rate and is given by

$$\lambda^e_{(R\_BOS)} = \lambda p_0 + \lambda(1 - p) \sum_{n=1}^{k-1} p_n$$

$$= \lambda(1 - p)(1 - p_k) + \lambda p p_0, \tag{3.1}$$

where ‘ $p_k$ ’ and ‘ $p_0$ ’ are given in (2.4) and (2.5) respectively.

Similarly

$$\lambda^e_{(R\_EOS)} = \lambda(1 - p)(1 - q_k) + \lambda p q_0, \tag{3.2}$$

where ‘ $q_k$ ’ and ‘ $q_0$ ’ are given in (2.9) and (2.10) respectively.

It is relevant to note here that unlike in the traditional M/M/1 model where it is important that  $\lambda < \mu$ , something similar need not hold in M/M/1/k model as customers arriving after maximum buffer size has been reached are turned back.

We have assumed that each customer has a random patience time following exp ( $\nu$ ). Clearly then the reneging rate of the system would depend on the state of the system as well as the reneging rule. The average reneging rate (avg rr) is a measure of average loss due to reneging. For the two rules of reneging, it is obtained as

$$Avgrr_{(R\_BOS)} = \sum_{n=2}^k (n - 1) \nu p_n$$

$$= \nu \{L_{R\_BOS} - (1 - p_0)\} \tag{3.3}$$

$$= \lambda - (\mu + \lambda p)(1 - p_0) - \lambda(1 - p)p_k.$$

$$Avgrr_{(R\_EOS)} = \sum_{n=1}^k n \nu q_n$$

$$= \nu L_{R\_EOS} \tag{3.4}$$

$$= \lambda - (\mu + \lambda p)(1 - q_0) - \lambda(1 - p)q_k.$$

To the system manager, customers who balk or renege represent business lost. In totality, customers are lost to the system in three ways, due to finite buffer, due to renegeing and due to balking. The management would like to know the proportion of total customers lost in order to have an idea of total business lost.

The mean rate at which customers are lost (under R\_BOS) is

$$\lambda - \lambda^e_{(R\_BOS)} + avgrr_{(R\_BOS)} = \lambda - \mu(1 - p_0). \quad (3.5)$$

Similarly the mean rate at which customers are lost (under R\_EOS) is

$$\begin{aligned} &\lambda - \lambda^e_{(R\_EOS)} + avgrr_{(R\_EOS)} \\ &= \lambda - \mu(1 - q_0). \end{aligned} \quad (3.6)$$

These rates help in the determination of proportion of customers lost which is of interest to the system manager as also an important measure of business lost. This proportion (under R\_BOS) is given by

$$\begin{aligned} &\frac{\lambda - \lambda^e_{(R\_BOS)} + avgrr_{(R\_BOS)}}{\lambda}, \\ &= 1 - \frac{\mu}{\lambda}(1 - p_0). \end{aligned} \quad (3.7)$$

and the proportion (under R\_EOS) is given by

$$\begin{aligned} &\frac{\lambda - \lambda^e_{(R\_EOS)} + avgrr_{(R\_EOS)}}{\lambda}, \\ &= 1 - \frac{\mu}{\lambda}(1 - q_0). \end{aligned} \quad (3.8)$$

The proportion of customers completing receipt of service can now be easily determined from the above proportion.

The customers who leave the system do not receive service. Consequently, only those customers who reach the service station constitute the actual load of the server. From the server's point of view, this provides a measure of the amount of work he has to do. We shall denote by  $\lambda^s$ , the rate at which customers reach the service station. Then under R\_BOS

$$\begin{aligned} \lambda^s_{(R\_BOS)} &= \lambda^e_{(R\_BOS)}(1 - \text{proportion of customers lost due to renegeing out of those joining the system}) \\ &= \lambda^e_{(R\_BOS)} \left\{ 1 - \sum_{n=2}^k (n-1)p_n / \lambda^e_{(R\_BOS)} \right\} \\ &= \lambda^e_{(R\_BOS)} - avgrr_{(R\_BOS)} \\ &= \mu(1 - p_0). \end{aligned}$$

In case of R\_EOS, one needs to recall that customers may renege even while being served and only those customers who renege from the queue will not constitute any work for the server. Then

$\lambda^s_{(R\_EOS)} = \lambda^e_{(R\_EOS)}(1 - \text{proportion of customers lost due to renegeing from the queue out of those joining the system})$

$$\begin{aligned} &= \lambda^e_{(R\_EOS)} \left\{ 1 - \sum_{n=2}^k (n-1)vq_n / \lambda^e_{(R\_EOS)} \right\} \\ &= \lambda^e_{(R\_EOS)} - v \{ L_{R\_EOS} - (1 - q_0) \} \\ &= \lambda^e_{(R\_EOS)} - vL_{q(R\_EOS)} \\ &= (\mu - v)(1 - q_0). \end{aligned}$$

**Sensitivity Analysis:**

It is interesting to examine and understand how server utilization varies in response to change in system parameters. The four system parameters of interest are  $\lambda, \mu, \nu, k$ . We place below the effect of change in these system parameters on server utilization. For this purpose, we shall follow the following notational convention in the rest of this section.

$p_n(\lambda, \mu, \nu, k)$  and  $q_n(\lambda, \mu, \nu, k)$  will denote the probability that there are ‘n’ customers in a system with parameters  $\lambda, \mu, \nu, k$  in steady state under R\_BOS and R\_EOS respectively.

It can be shown that

i) If  $\lambda_1 > \lambda_0$  then

$$\frac{p_0(\lambda_1, \mu, \nu, k)}{p_0(\lambda_0, \mu, \nu, k)} < 1$$

$$\Rightarrow \frac{(\lambda_0 - \lambda_1)}{\mu} + \frac{(1-p)}{\mu(\mu + \nu)}(\lambda_0^2 - \lambda_1^2) + \dots + \frac{(1-p)^{k-1}}{\mu(\mu + \nu) \dots \{\mu + (k-1)\nu\}}(\lambda_0^k - \lambda_1^k) < 0$$

which is true and hence  $p_0 \downarrow$  as  $\lambda \uparrow$ .

ii) If  $\mu_1 > \mu_0$  then

$$\frac{p_0(\lambda, \mu_1, \nu, k)}{p_0(\lambda, \mu_0, \nu, k)} > 1$$

$$\Rightarrow \lambda \left( \frac{1}{\mu_0} - \frac{1}{\mu_1} \right) + \lambda^2 (1-p) \left\{ \frac{1}{\mu_0(\mu_0 + \nu)} - \frac{1}{\mu_1(\mu_1 + \nu)} \right\} + \dots +$$

$$\lambda^k (1-p)^{k-1} \left\{ \frac{1}{\mu_0(\mu_0 + \nu) \dots \{\mu_0 + (k-1)\nu\}} - \frac{1}{\mu_1(\mu_1 + \nu) \dots \{\mu_1 + (k-1)\nu\}} \right\} > 0$$

which is true and hence  $p_0 \uparrow$  as  $\mu \uparrow$ .

iii) If  $\nu_1 > \nu_0$  then

$$\frac{p_0(\lambda, \mu, \nu_1, k)}{p_0(\lambda, \mu, \nu_0, k)} > 1$$

$$= \lambda \left( \frac{1}{\mu} - \frac{1}{\mu} \right) + \lambda^2 (1-p) \left\{ \frac{1}{\mu(\mu + \nu_0)} - \frac{1}{\mu(\mu + \nu_1)} \right\} + \dots +$$

$$\lambda^k (1-p)^{k-1} \left\{ \frac{1}{\mu(\mu + \nu_0) \dots \{\mu + (k-1)\nu_0\}} - \frac{1}{\mu(\mu + \nu_1) \dots \{\mu + (k-1)\nu_1\}} \right\} > 0$$

which is true and hence  $p_0 \uparrow$  as  $\nu \uparrow$ .

iv) If  $k_1 > k_0$  then

$$\frac{p_0(\lambda, \mu, \nu, k_1)}{p_0(\lambda, \mu, \nu, k_0)} < 1$$

$$\Rightarrow \sum_{n=1}^{k_0} \frac{\lambda^n (1-p)^{n-1}}{\prod_{r=1}^n \{\mu + (r-1)\nu\}} - \sum_{n=1}^{k_1} \frac{\lambda^n (1-p)^{n-1}}{\prod_{r=1}^n \{\mu + (r-1)\nu\}} < 0$$

which is true and hence  $p_0 \downarrow$  as  $k \uparrow$ .

The following can similarly be shown.

- v)  $q_0 \downarrow$  as  $\lambda \uparrow$
- vi)  $q_0 \uparrow$  as  $\mu \uparrow$
- vii)  $q_0 \uparrow$  as  $\nu \uparrow$
- viii)  $q_0 \downarrow$  as  $k \uparrow$

Managerial implications of the above results are obvious.

**Numerical Example:**

To illustrate the use of our results, we apply them to a queuing problem. We quote below an example from Allen (2005, page 267 and 273).

‘Traffic to a message switching centre for Extraterrestrial Communications Corporation arrives in a random pattern (remember that ‘random pattern’ means exponential interarrival time) at an average rate of 240 messages per minute. The line has a transmission rate of 800 characters per second. The message length distribution (including control characters) is approximately exponential with an average length of 176 characters. Calculate the principal statistical measures of system performance assuming that a very large number message buffers is provided.’

‘Suppose, however, that it is desired to provide only the minimum number of messages buffers required to guarantee that  $p_k < 0.005$

How many buffers should be provided?’

This is a design problem. Here  $\lambda = 4/\text{sec}$  and  $\mu = 4.55/\text{sec}$ . As required by the switching centre, we examine the minimum number of message buffers with different choices of k. Though not explicitly mentioned, it is necessary to assume renegeing and balking. Balking because in telecommunication systems, it is known that an incoming message that sees a workload may be admitted in to the system with a certain probability and rejected otherwise. Rejection implies balking. We shall assume that the balking probability is ‘p’ (Boxma et al. 2008, has analyzed with a similar motivation).

Further we assume R\_BOS because in telecommunication systems it is also known that messages usually have some real time constraints within which the message has to be processed. Messages received after the deadline is considered obsolete and discarded. This can be seen as renegeing (Movaghar, 1998 and Boots and Tijms, 1999). Let us assume alternative possible Markovian renegeing rates of  $\nu = 0.1/\text{sec}$ ,  $\nu = 0.03333/\text{sec}$ . Put differently, these rates imply that a message joining the buffer will be alive if processing commences within 10sec and 30sec respectively on the average. We further assume that balking rate is independent of state and is taken as  $p = 0.001$  (one in 1000 message).

Various performance measures of interest computed under different scenarios are given in Table 1 and 2. These measures were arrived at using a FORTRAN 77 program coded by the authors. Different choices of k were considered. Results relevant with regard to the requirement that the switching centre should provide only the minimum number of message buffers to guarantee should  $p_k < 0.005$  are presented in the tables. (All rates in the following tables are per second).

**Table 1:-** Performance Measures assuming  $\lambda=4, \mu=4.55, \nu=0.1$  and  $p=0.001$ .

Performance Measure	Size of minimum number of message buffers		
	k=13	k=14	k=15
$p_k$	0.00704	0.00478	.00321
$\lambda^s$ (i.e. arrival rate of customers reaching service station)	3.71285	3.71686	3.71953
Effective mean arrival rate( $\lambda^e$ )	3.96859	3.97759	3.98392
Fraction of time server is idle ( $p_0$ )	0.18399	0.18311	0.18252

Average length of queue	2.55735	2.60725	2.64388
Average length of system	3.37336	3.42425	3.46136
Mean renegeing rate	0.25573	0.26074	0.26439
Mean rate of customers lost	0.28715	0.28314	0.28047
Proportion of customers loss due to renegeing, balking and finite buffer	0.07179	0.07078	0.07012

**Table 2:-** Performance Measures assuming  $\lambda=4$ ,  $\mu=4.55$ ,  $v=0.03333$  and  $p=0.001$ .

Performance Measure	Size of minimum number of message buffers		
	k=18	k=19	k=20
$p_k$	0.00505	0.00390	.00299
$\lambda^s$ (i.e. arrival rate of customers reaching service station)	3.85368	3.85639	3.85848
Effective mean arrival rate( $\lambda^e$ )	3.97644	3.98102	3.98463
Fraction of time server is idle ( $p_0$ )	0.15304	0.15244	0.15198
Average length of queue	3.68314	3.73899	3.78476
Average length of system	4.53009	4.58655	4.63277
Mean renegeing rate	0.12276	0.12462	0.12615
Mean rate of customers lost	0.14632	0.14360	0.14152
Proportion of customers loss due to renegeing, balking and finite buffer	0.03658	0.03590	0.03538

In case the renegeing behavior follows exp (0.1) distribution, it appears from the tables 1 that an ideal choice of k could be 14 with  $p_k=0.00478$ . If the renegeing distribution is exp (0.03333), then k=19 appears to be close to the switching centre with  $p_k=.00390$  (table 2).

A few interesting observations can be made from the above tables.

1) The rate of arrival into the system is 4message/sec whereas the actual load of the server ( $\lambda^s$ ) is 3.71686 (from table 1 case of k=14). Thus  $4-3.71686=0.26398$  message/sec are being lost. As a proportion of all message arriving at the centre this amounts to 0.07078 ( $=0.28314/4$ ) proportion of lost messages. This can be confirmed from the last row of the table.

2) Since  $\lambda^s=3.71686$ , the centre is required to process 3.71686 messages/sec when its capacity is 4.55 messages/sec. The centre is therefore idle for 0.18311( $=(4.55-3.71686)/4.55$ ) proportion of time. This corroborates  $p_0$  in table 1 under k=14.

Similar observations can be made for other cases in table 2.

### Conclusion:-

The analysis of a single server finite buffer Markovian queuing system with state-independent balking and Markovian renegeing has been discussed. Closed form expressions of number of performance measures have been presented. Many of these could be of direct interest to the system manger. To study the change in the system corresponding to change in system parameters, sensitivity analysis has also been presented. A numerical example with design connotations has been discussed to demonstrate results derived. An obvious extension would be the finite buffer multiserver case. Work towards that end is ongoing.

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### Appendix:

#### Derivation of $P'(1)$ under R\_BOS:

From equation (2.3) we have

$$\lambda(1-p)p_{n-1} + (\mu + n\nu)p_{n+1} = \lambda(1-p)p_n + \{\mu + (n-1)\nu\}p_n \quad n = 2,3,\dots,k-1,$$

Now multiplying both sides of the equation by  $s^n$  and summing over  $n$

$$\begin{aligned}
 & \lambda s \sum_{n=2}^{k-1} (1-p)p_{n-1}s^{n-1} + \frac{1}{s} \sum_{n=2}^{k-1} (\mu + n\nu)p_{n+1}s^{n+1} = \lambda \sum_{n=2}^{k-1} (1-p)p_n s^n + \sum_{n=2}^{k-1} \{\mu + (n-1)\nu\} p_n s^n \\
 \Rightarrow & \lambda s(1-p) \sum_{n=2}^{k-1} p_{n-1}s^{n-1} - \lambda(1-p) \sum_{n=2}^{k-1} p_n s^n = \sum_{n=2}^{k-1} \{\mu + (n-1)\nu\} p_n s^n - \frac{1}{s} \sum_{n=2}^{k-1} (\mu + n\nu)p_{n+1}s^{n+1} \\
 \Rightarrow & \lambda s(1-p) \{p_1 s^1 + p_2 s^2 + p_3 s^3 + \dots + p_{k-2} s^{k-2}\} - \lambda(1-p) \{p_2 s^2 + p_3 s^3 + \dots + p_{k-1} s^{k-1}\} \\
 = & \left[ (\mu + \nu)p_2 s^2 + (\mu + 2\nu)p_3 s^3 + \dots + \{\mu + (k-2)\nu\} p_{k-1} s^{k-1} \right] - \frac{1}{s} \left[ (\mu + 2\nu)p_3 s^3 + (\mu + 3\nu)p_4 s^4 + \dots + \{\mu + (k-1)\nu\} p_k s^k \right] \\
 \Rightarrow & \lambda s(1-p) \{p(s) - p_0 - p_{k-1} s^{k-1} - p_k s^k\} - \lambda(1-p) \{p(s) - p_0 - p_1 s - p_k s^k\} \\
 = & \left\{ \mu(p_2 s^2 + p_3 s^3 + \dots + p_{k-1} s^{k-1}) + \nu s(p_2 s + 2p_3 s^2 + \dots + (k-2)p_{k-1} s^{k-2}) \right\} \\
 & - \frac{1}{s} \left\{ \mu(p_3 s^3 + p_4 s^4 + \dots + p_k s^k) + \nu s(2p_3 s^2 + 3p_4 s^3 + \dots + (k-1)p_k s^{k-1}) \right\} \\
 \Rightarrow & \lambda s(1-p) \{p(s) - p_0 - p_{k-1} s^{k-1} - p_k s^k\} - \lambda(1-p) \{p(s) - p_0 - p_1 s - p_k s^k\} \\
 = & \left[ \mu \{p(s) - p_0 - p_1 s - p_k s^k\} + \nu s \{p'(s) - p_1 - k p_k s^{k-1}\} - \nu \{p(s) - p_0 - p_1 s - p_k s^k\} \right] \\
 & - \frac{1}{s} \left[ \mu \{p(s) - p_0 - p_1 s - p_2 s^2\} + \nu s \{p'(s) - p_1 - 2p_2 s\} - \nu \{p(s) - p_0 - p_1 s - p_2 s^2\} \right] \\
 \Rightarrow & \lambda s(1-p)p(s) - \lambda s(1-p)p_0 - \lambda(1-p)p_{k-1} s^k - \lambda s(1-p)p_k s^k - \lambda(1-p)p(s) + \lambda(1-p)p_0 + \lambda s(1-p)p_1 + \lambda(1-p)p_k s^k \\
 = & \mu p(s) - \mu p_0 - \mu p_1 s - \mu p_k s^k + \nu s p'(s) - \nu s p_1 - k \nu p_k s^k - \nu p(s) + \nu p_0 + \nu p_1 s + \nu p_k s^k - \frac{\mu}{s} p(s) + \frac{\mu}{s} p_0 + \mu p_1 \\
 & + \mu p_2 s - \nu p'(s) + \nu p_1 + 2\nu p_2 s + \frac{\nu}{s} p(s) - \frac{\nu}{s} p_0 - \nu p_1 - \nu p_2 s \\
 \Rightarrow & \lambda s(1-p)p(s) - \lambda s(1-p)p_0 - \lambda(1-p) \frac{\{\mu + (k-1)\nu\}}{\lambda(1-p)} p_k s^k - \lambda s(1-p)p_k s^k - \lambda(1-p)p(s) + \lambda(1-p)p_0 \\
 & + \lambda s(1-p)p_1 + \lambda(1-p)p_k s^k \\
 = & \mu p(s) - \mu p_0 - \mu p_1 s - \mu p_k s^k + \nu s p'(s) - \nu s p_1 - k \nu p_k s^k - \nu p(s) + \nu p_0 + \nu p_1 s + \nu p_k s^k - \frac{\mu}{s} p(s) + \frac{\mu}{s} p_0 + \mu p_1 \\
 & + \mu p_2 s - \nu p'(s) + \nu p_1 + 2\nu p_2 s + \frac{\nu}{s} p(s) - \frac{\nu}{s} p_0 - \nu p_1 - \nu p_2 s \\
 \Rightarrow & \nu p'(s) - \nu s p'(s) = p(s) \left( \lambda(1-p) - \lambda s(1-p) + \mu - \frac{\mu}{s} + \frac{\nu}{s} - \nu \right) + \{\mu + (k-1)\nu - \mu - k\nu + \nu + \lambda s(1-p) - \lambda(1-p)\} p_k s^k \\
 & - \mu p_0 - \mu s \frac{\lambda}{\mu} p_0 + \nu p_0 + \frac{\mu}{s} p_0 + \mu \frac{\lambda}{\mu} p_0 - \frac{\nu}{s} p_0 + (\mu + \nu) \frac{\lambda^2(1-p)}{\mu(\mu + \nu)} s p_0 + \lambda s(1-p)p_0 - \lambda(1-p)p_0 \\
 & - \lambda s(1-p) \frac{\lambda}{\mu} p_0 \\
 \Rightarrow & \nu(1-s)p'(s) = \frac{\{\nu(1-s) - \mu(1-s) + \lambda s(1-p)(1-s)\}}{s} p(s) - \lambda(1-p)(1-s)p_k s^k - \frac{\mu(1-s)}{s} p_0 - \frac{\nu(1-s)}{s} p_0 \\
 & + \lambda(1-s)p_0(1-1+p)
 \end{aligned}$$

$$\Rightarrow v(1-s)p'(s) = \frac{(1-s)\{v - \mu + \lambda s(1-p)\}}{s} p(s) - \lambda(1-p)(1-s)p_k s^k + \frac{(1-s)(\mu - v + \lambda sp)}{s} p_0$$

$$\Rightarrow p'(s) = \frac{\{\lambda s(1-p) - \mu + v\}}{sv} p(s) - \frac{\lambda}{v}(1-p)p_k s^k + \frac{(\mu - v + \lambda sp)}{sv} p_0$$

Now

$$\lim_{s \rightarrow 1-} p'(s) = \lim_{s \rightarrow 1-} \left[ \frac{\{\lambda s(1-p) - \mu + v\}}{sv} p(s) - \frac{\lambda}{v}(1-p)p_k s^k + \frac{(\mu - v + \lambda sp)}{sv} p_0 \right]$$

$$s \rightarrow 1- \quad s \rightarrow 1-$$

$$\begin{aligned} \Rightarrow p'(1) &= \frac{\{\lambda(1-p) - \mu + v\}}{v} - \frac{\lambda}{v}(1-p)p_k + \frac{(\mu - v + \lambda p)}{v} p_0 \\ &= \frac{\lambda}{v} - \frac{(\mu - v + \lambda p)(1-p_0)}{v} - \frac{\lambda(1-p)}{v} p_k \\ &= \frac{\lambda - (\mu - v + \lambda p)(1-p_0) - \lambda(1-p)p_k}{v} \end{aligned} \tag{7.1.1}$$

**Derivation of Q'(1) under R\_EOS:**

From equation (2.8) we have

$$\lambda(1-p)q_{n-1} + \{\mu + (n+1)v\}q_{n+1} = \lambda(1-p)q_n + (\mu + nv)q_n \quad n=2,3,\dots,k-1.$$

Multiplying both side of this equation by  $s^n$  and summing over  $n$  from 2 to  $k-1$  we get

$$\lambda s(1-p) \sum_{n=2}^{k-1} q_{n-1} s^{n-1} - \lambda(1-p) \sum_{n=2}^{k-1} q_n s^n = \sum_{n=2}^{k-1} (\mu + nv)q_n s^n - \frac{1}{s} \sum_{n=2}^{k-1} \{\mu + (n+1)v\}q_{n+1} s^{n+1}$$

Proceeding similar to the previous section 7.1, we obtain,

$$\Rightarrow Q'(1) = \frac{\lambda - (\mu + \lambda p)(1-p_0) - \lambda(1-p)q_k}{v}. \tag{7.2.1}$$