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RESEARCH ARTICLE

SPHERICAL SYMMETRIC SOLITONS OF SPINOR FIELD IN GRAVITATIONAL THEORY

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Abstract

This paper deals with soliton-like solutions as model in order to describe the configurations of elementary particles through the interaction of their fields in general relativity. To this end, we have obtained exact spherical symmetric soliton-like solutions to the nonlinear spinor field equations, taking into account the proper gravitational field of elementary particles. The nonlinear terms in the spinor field lagrangian are given by an arbitrary function L_N depending on the invariant $I_T = S^2 - P^2$ with $S = \bar{\psi}\psi$ and $P = i\bar{\psi}\gamma^5\psi$. It is shown that, under certain choice of the nonlinear terms in the spinor lagrangian, the solutions are regular with a localized energy density, limited total energy only if $m=0$ (m is the mass parameter in the spinor field equations). In addition, the total charge and the total spin are bounded. The solutions to Heisenberg-Ivanenko nonlinear equation have been also obtained. Let us emphasize that Heisenberg-Ivanenko nonlinear spinor field equation possesses soliton-like solutions. Later, the influence of the nonlinear terms in the formation of the fields configurations with limited total energy have been examined. We noted that, in linear case, the obtained solutions are regular and having an unlimited energy density. Nevertheless, exact solutions, including soliton-like configurations exist in flat space-time.

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Introduction:-

At present, the nonlinear generalization of classical field theory remains one of the possible ways to overcome the difficulties of the theory, which considers elementary particles as mathematical points. In this approach, elementary particles are modeled by soliton-like solutions of corresponding nonlinear equations [1]. The soliton is a regular solution of nonlinear differential equations with localized energy density, a bounded total energy and stable. It is widely present in many branches in pure science and used for different purposes. The concept of soliton is thoroughly dealt in a series of papers. A.H. Taub has defined the characteristics of inhomogeneous plane-symmetric metric of the space-time in [2, 3]. Thus, considering plane-symmetric metric of the space-time under the form:

$$ds^2 = e^{2\gamma} dt^2 - e^{2\alpha} dx^2 - e^{2\beta} (dy^2 + dz^2)$$

where the metric functions $g_{00} = e^{2\gamma}$, $g_{11} = -e^{2\alpha}$ and $g_{22} = g_{33} = -e^{2\beta}$ are time independent, plane-symmetric solutions have been obtained in a series of articles [4,5]. The authors, in all these activities, investigated the

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influence of nonlinear terms in the nonlinear fields equations. They also examined the role of the proper gravitational field of elementary particles by solving Einstein's and spinor field equations in the flat space-time. Let us emphasize that the obtained solutions are singular to the metric considered because the components of Riemann-Christoffel tensor are limited but the total charge Q and the total spin S_1 diverge. Later, spherical symmetric soliton-like solutions to the nonlinear spinor field equations in General Relativity Theory have been obtained in series remarkable papers [6, 7, 8]. It is demonstrated that the solutions obtained possess a finite value of total charge and total spin. These results confirm the importance of the metric and its geometric properties in the configuration of the structure of elementary particles by soliton model in General Relativity Theory. The symmetries of the gravitational field plays an important role in general relativity. Its importance in the theory of general relativity has been dealt with by Katzin, Lavine and Davis in a series of papers. An excellent review on fundamental symmetry of the space-time of the general relativity defined by the vanishing Lie derivative of the Riemann curvature tensor may be found in [9]. Then, for details literature on groups of curvature collineation in Riemannian space-times, which admit fields of parallel, refer to [10]. As for the applications of Lie derivative to symmetries, geodesic mappings and first integrals in Riemannian spaces, see [11] and references therein.

The purpose of the paper is to obtain the spherical symmetric solitons in microcosmos of spinor field in general relativity, taking into account the own gravitational field of elementary particles.

This paper is organized as follows. First, in section 1, we briefly did the literature review on soliton. Section 2 is intended for the model and fundamental equations. Thus, applying the variational principle and usual algebraic manipulations, we established the fundamental equations. The nonlinearity in the spinor field lagrangian is given by an arbitrary function of the invariant $I_T = I_S - I_P$ where $I_S = (\bar{\psi}\psi)^2$ and $I_P = (i\bar{\psi}\gamma^5\psi)^2$. Section 3 deals with the results through the fundamental solutions. The section 4 addresses to the discussion. To this end, we have chosen a concrete form of nonlinear terms in the lagrangian density. In the same section, we proved the contribution of the nonlinearity in the configuration of the geometrical structure of elementary particles. Lastly, Section 5 presents concluding remarks and future work.

Lagrangian, Metric, Basic Fields Equations:-

In the present analysis, the lagrangian of the self-consistent system of spinor and gravitational fields is defined as follows:

$$L = \frac{R}{2k} + \frac{i}{2} (\bar{\psi}\gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi) - m \bar{\psi} \psi + L_N \quad (1)$$

where R is the scalar curvature, $k = \frac{8\pi G}{c^4}$ is Einstein's gravitational constant, G is Newton's gravitational constant and c is the speed of light in the vacuum. $L_N = H(I_T)$ is an arbitrary function of the invariant function I_T corresponding to the real bilinear form $I_T = T_{\mu\nu} T^{\mu\nu} = (\bar{\psi} \sigma^{\mu\nu} \psi) g_{\mu\alpha} g_{\nu\beta} (\bar{\psi} \sigma^{\alpha\beta} \psi)$. L_N is the nonlinear term of the spinor lagrangian characterizing the self-interaction of the spinor field. The following paragraph deals with the metric of the space-time used in this manuscript.

In this paper, instead of the static plane-symmetric metric chosen in [12], we opt for the static spherical symmetric metric defined in the pseudo-riemannian varieties by the following expression:

$$ds^2 = e^{2\gamma} dt^2 - e^{2\alpha} d\xi^2 - e^{2\beta} (d\theta^2 + \sin^2(\theta) d\phi^2). \quad (2)$$

The signature of the metric is (+1, -1, -1, -1) and the velocity of light is chosen to be unity in natural unities ($c=1$). The metric functions α, β and γ are time and coordinates angular θ and ϕ independent. They depend only on the variable $\xi = \frac{1}{r}$ where r stands for the radial component of the static spherical symmetric metric defined above. They obey the harmonic coordinates condition in the form:

$$\alpha(\xi) = 2\beta(\xi) + \gamma(\xi). \quad (3)$$

The following paragraph deals with the Einstein's equations, the spinor field equations and the energy-momentum tensor. From the lagrangian (1), through the variational principle and usual algebraic manipulations, we obtain the components of Einstein's tensor and the spinor field equations for the functions ψ and $\bar{\psi}$ in the metric (2) under the coordinate condition (3) as follows:

$$\begin{aligned}
 G_0^0 &= e^{-2\alpha}(2\beta'' - 2\gamma'\beta' - \beta'^2) - e^{-2\beta} = -\kappa T_0^0 \\
 (4)G_1^1 &= e^{-2\alpha}(2\beta'\gamma' + \beta'^2) - e^{-2\beta} = -\kappa T_1^1, \\
 G_2^2 &= e^{-2\alpha}(\beta'' + \gamma'' - 2\gamma'\beta' - \beta'^2) = -\kappa T_2^2, \\
 G_2^2 &= G_3^3, \quad T_2^2 = T_3^3. (7)
 \end{aligned}
 \tag{5}$$

As for the nonlinear spinor field equations, we have:

$$i\gamma^\mu \nabla_\mu \psi - m\psi + 2S \frac{\partial H}{\partial I_T} \psi - 2iP \frac{\partial H}{\partial I_T} \gamma^5 \psi = 0 \tag{8}$$

$$i\nabla_\mu \bar{\psi} \gamma^\mu + m\bar{\psi} - 2S \frac{\partial H}{\partial I_T} \bar{\psi} + 2iP \frac{\partial H}{\partial I_T} \gamma^5 \bar{\psi} = 0 \tag{9}$$

The corresponding energy-momentum tensor of the spinor fields:

$$T_\mu^\nu = \frac{1}{4} g^{\mu\rho} (\bar{\psi} \gamma_\mu \nabla_\nu \psi + \bar{\psi} \gamma_\nu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma_\nu \psi - \nabla_\nu \bar{\psi} \gamma_\mu \psi) - \delta_\mu^\nu L_{Sp}. \tag{10}$$

From the spinor field equations(8) and (9), the expression of L_{Sp} may be written under the form:

$$\begin{aligned}
 L_{Sp} &= \frac{1}{2} \bar{\psi} (i\gamma^\mu \nabla_\mu \psi - m\psi) - \frac{1}{2} (i\nabla_\mu \bar{\psi} \gamma^\mu + m\bar{\psi}) \psi + H(S, P), \\
 &= -2I_S \frac{\partial H}{\partial I_S} - 2I_P \frac{\partial H}{\partial I_P} + H(S, P) = -2I_T \frac{\partial H}{\partial I_T} + H(S, P).
 \end{aligned}
 \tag{11}$$

With the expression (11), the nontrivial components of the energy-momentum tensor are:

$$T_0^0 = T_2^2 = T_3^3 = -L_{Sp} = 2I_T \frac{\partial H}{\partial I_T} - H(S, P). \tag{12}$$

$$T_1^1 = \frac{1}{2} (\bar{\psi} \gamma^1 \nabla_1 \psi - \nabla_1 \bar{\psi} \gamma^1 \psi) + 2I_T \frac{\partial H}{\partial I_T} - H(S, P). \tag{13}$$

In (8)-(10), ∇_μ denotes the covariant derivative of spinor field. As defined in [13], ∇_μ is connected to the functions ψ and $\bar{\psi}$ as follows:

$$\nabla_\mu \psi = \frac{\partial \psi}{\partial \xi^\mu} - \Gamma_\mu \psi \quad \text{or} \quad \nabla_\mu \bar{\psi} = \frac{\partial \bar{\psi}}{\partial \xi^\mu} + \Gamma_\mu \tag{14}$$

where $\Gamma_\mu(\xi)$ denotes the spinor affine connection matrices. The general expression of $\Gamma_\mu(\xi)$ is given by the following equality:

$$\Gamma_\mu(\xi) = \frac{1}{4} g_{\rho\mu} (\partial_\mu e_\sigma^b e_a^p - \Gamma_{\mu\sigma}^p) \gamma^\delta \gamma^\sigma \tag{15}$$

In (15) γ^δ denotes the Dirac's matrices in curved space-time. It is linked to the Dirac's matrices $\bar{\gamma}^\delta$ in flat space-time by :

$$\gamma^0(\xi) = e^{-\gamma} \bar{\gamma}^0, \quad \gamma^1(\xi) = e^{-\alpha} \bar{\gamma}^1, \quad \gamma^2(\xi) = e^{-\beta} \bar{\gamma}^2 \quad \text{and} \quad \gamma^3(\xi) = \frac{e^{-\beta} \bar{\gamma}^3}{\sin\theta} \tag{16}$$

The Dirac's matrices $\bar{\gamma}^\delta$ in flat space-time are chosen as in [14, 15].

The relations (15) and (16) lead to :

$$\Gamma_0 = -\frac{1}{2} e^{-2\beta} \bar{\gamma}^0 \bar{\gamma}^1 \gamma', \quad \Gamma_1 = 0, \quad \Gamma_2 = \frac{1}{2} e^{-\beta} \bar{\gamma}^2 \bar{\gamma}^1 \beta', \quad \Gamma_3 = \frac{1}{2} (e^{-\beta} \bar{\gamma}^3 \bar{\gamma}^1 \beta' \sin\theta + \bar{\gamma}^3 \bar{\gamma}^2 \cos\theta) \tag{17}$$

Accordingg to the Einstein's convention, from (16) and (17), we have :

$$\gamma^\mu \Gamma_\mu = -\frac{1}{2} (e^{-\alpha} \alpha' \bar{\gamma}^1 + \bar{\gamma}^2 e^{-\beta} \cot\theta). \tag{18}$$

Substituting (14) and (18) into (8) and (9), the spinor field equations may be rewritten as follows

$$ie^{-\alpha} \gamma^{-1} \left(\partial_\xi + \frac{1}{2} \alpha' \right) \psi + \frac{1}{2} \bar{\gamma}^2 e^{-\beta} \psi \cot\theta - (m - D)\psi - iG(S, P) \gamma^5 \psi = 0, \tag{19}$$

$$ie^{-\alpha} \gamma^{-1} \left(\partial_\xi + \frac{1}{2} \alpha' \right) \bar{\psi} + \frac{1}{2} \bar{\gamma}^2 e^{-\beta} \bar{\psi} \cot\theta + (m - D)\bar{\psi} + iG(S, P) \gamma^5 \bar{\psi} = 0, \tag{20}$$

where

$$D(S, P) = 2S \frac{dH}{dI_T}, \quad G(S, P) = 2P \frac{dH}{dI_T} \tag{21}$$

We get the set following of equations from (19), setting $\psi(\xi) = V_\delta(\xi); \delta = 1, 2, 3$ and 4:

$$V_4 + \frac{1}{2} \alpha' V_4 - \frac{1}{2} e^{\alpha-\beta} V_4 \cot\theta + ie^\alpha (m-D) V_1 + Ge^\alpha V_3 = 0 \tag{22}$$

$$V_3 + \frac{1}{2} \alpha' V_3 + \frac{1}{2} e^{\alpha-\beta} V_3 \cot\theta + ie^\alpha (m-D) V_2 + Ge^\alpha V_4 = 0 \tag{23}$$

$$V_2 + \frac{1}{2} \alpha' V_2 - \frac{1}{2} e^{\alpha-\beta} V_2 \cot\theta - ie^\alpha (m-D) V_3 - Ge^\alpha V_1 = 0 \tag{24}$$

$$V_1 + \frac{1}{2} \alpha' V_1 - \frac{1}{2} e^{\alpha-\beta} V_1 \cot\theta - ie^\alpha (m-D) V_4 - Ge^\alpha V_2 = 0 \tag{25}$$

Let us emphasize that the resolution of the system of equations (22)-(25) consists to determine $D(S, P)$, $G(S, P)$, S and P as functions of $e^{\alpha(\xi)}$. The previous set of equations has allowed to define the system of equations of the invariant functions $S = \bar{\psi} \psi$, $P = i \bar{\psi} \gamma^5 \psi$ and $R = \bar{\psi} \gamma^5 \bar{\gamma}^1 \psi$. The following section deals with the results.

Results:-

By summing the set of equations (22)-(25), we get:

$$S' + \alpha S - 2Ge^{\alpha R} = 0 \tag{26}$$

$$R' + \alpha R + 2(m-D)e^{\alpha P} - 2Ge^{\alpha S} = 0 \tag{27}$$

$$P' + \alpha P + 2(m-D)e^{\alpha R} = 0 \tag{28}$$

The system of equations (26)-(28) has general solution:

$$P^2 - R^2 + S^2 = be^{-2\alpha(\xi)} = -\frac{b}{g_{11}}, \tag{29}$$

b being a constant.

Let us study a massless particle case (m=0), without losing of the generality. In order to simplify the set of equations (26)-(28), let us consider:

$$P_0(\xi) = e^{\alpha P}(\xi); \quad S_0(\xi) = e^{\alpha S}(\xi) \quad \text{and} \quad R_0(\xi) = e^{\alpha R}(\xi) \tag{30}$$

We find

$$S'_0 - 2Ge^{\alpha R_0} = 0 \tag{31}$$

$$R'_0 - 2De^{\alpha P_0} - 2Ge^{\alpha S_0} = 0 \tag{32}$$

$$P'_0 - 2De^{\alpha R_0} = 0 \tag{33}$$

The first integral of the system equations (31)-(33) is :

$$P_0^2 - R_0^2 + S_0^2 = b_0, \tag{34}$$

b₀ is integration constant

Multiplying respectively (31) and (33) by D(S, P) and G(S,P). Then, by summing the results, we obtain

$$D(S, P) S'_0 = G(S, P) P'_0 \tag{35}$$

From (21) and (30), the expression (35) leads:

$$S_0 S'_0 = P_0 P'_0 \tag{36}$$

The first integral of the equation (36) is:

$$S_0^2 - P_0^2 = b_1, \quad I_T(\xi) = S^2 - P^2 = b_1 e^{-2\alpha(\xi)} \tag{37}$$

From (34) and (37), we get :

$$P_0^2 = S_0^2 - b_1, \quad R_0^2 = 2S_0^2 - b_1 - b_0 \tag{38}$$

Taking into account (38), the equation (31) becomes:

$$\int \frac{dS_0}{\sqrt{(2S_0^2 - b_1 - b_0)(S_0^2 - b_1)}} = 4 \int \frac{dH}{dI_T} d\xi + b_2, \quad b_2 = \text{const.} \tag{39}$$

The obtained solution may be expressed by elliptic functions in the general case of the arbitrary constants b₀ and b₁. In the sequel, we study the particular case where b₀ + b₁ = 0. To this end, we consider b₁ as a negative or positive constant.

- For b₁ = -a₀² < 0, we have :

$$S_0(\xi) = -\frac{a_0}{\sinh[a_0\sqrt{2}\Lambda(\xi)]}, \quad S(\xi) = -\frac{a_0}{\sinh[a_0\sqrt{2}\Lambda(\xi)]} e^{-\alpha(\xi)} \tag{40}$$

with

$$\Lambda(\xi) = 4 \int \frac{dH}{dI_T} d\xi + b_2, \quad b_2 = \text{const.} \tag{41}$$

Then, we deduce that :

$$P_0(\xi) = a_0 \coth[a_0\sqrt{2}\Lambda(\xi)], \quad P(\xi) = a_0 \coth[a_0\sqrt{2}\Lambda(\xi)] e^{-\alpha(\xi)} \tag{42}$$

The expression of the functions D(S, P) and G(S, P) are given by:

$$D(S, P) = -\frac{2a_0}{\sinh[a_0\sqrt{2}\Lambda(\xi)]} e^{-\alpha(\xi)} \frac{dH}{dI_T} \tag{43}$$

$$G(S, P) = 2a_0 \coth[a_0\sqrt{2}\Lambda(\xi)] e^{-\alpha(\xi)} \frac{dH}{dI_T} \tag{44}$$

- For b₁ = a₁ > 0, we obtain directly:

$$S_0(\xi) = -\frac{a_1}{\cosh[a_1\sqrt{2}\Lambda(\xi)]}, \quad S(\xi) = -\frac{a_1}{\cosh[a_1\sqrt{2}\Lambda(\xi)]} e^{-\alpha(\xi)} \tag{45}$$

$$P_0(\xi) = ia_1 \tanh[a_1\sqrt{2}\Lambda(\xi)], \quad P(\xi) = ia_1 \tanh[a_1\sqrt{2}\Lambda(\xi)] e^{-\alpha(\xi)} \tag{46}$$

In these conditions, we get the general forms of the functions D(S, P) and G(S, P), depending on the arbitrary function H(I_T) as follows:

$$D(S, P) = -\frac{2a_1}{\cosh[a_1\sqrt{2}\Lambda(\xi)]} e^{-\alpha(\xi)} \frac{dH}{dI_T} \quad (47)$$

$$G(S, P) = 2a_1 \tanh[a_1\sqrt{2}\Lambda(\xi)] e^{-\alpha(\xi)} \frac{dH}{dI_T} \quad (48)$$

Let us remark that knowing the explicit analytic form of the arbitrary function $H(I_T)$, we can determine the exact form of the function $\Lambda(\xi)$ and write $S(\xi)$, $P(\xi)$, $D(S,P)$ and $G(S,P)$ as functions of $e^{\alpha(\xi)}$.

Taking into account the spinor field in the form (21) and the conjugate one, we obtain the following expression for the component T_1^1 of the energy-momentum metric tensor from (13):

$$T_1^1 = mS(\xi) - H(I_T) \quad (49)$$

The paragraph to follow deals with the resolution of Einstein's equations. In this purpose, with the equality $T_0^0 = T_2^2$, the difference $\binom{0}{0} - \binom{2}{2}$ leads to the following expression:

$$\beta'' - \gamma'' = e^{2\beta+2\gamma} \quad (50)$$

The equation (50) can be transformed into a Liouville equation. For details, see [16] and references therein.

The general solutions of the equation (50) are:

$$\beta(\xi) = \frac{M}{4} \left(1 + \frac{2}{A}\right) \ln \frac{M}{AT^2(h, \xi + \xi_1)} = \left(1 + \frac{2}{A}\right) \gamma(\xi) \quad (51)$$

$$\gamma(\xi) = \frac{M}{4} \ln \frac{M}{AT^2(h, \xi + \xi_1)} \quad (52)$$

A and M being unknown integration constants.

The function T possesses the following form:

$$T(h, \xi + \xi_1) = \begin{cases} \frac{1}{h} \sinh[h(\xi + \xi_1)], & h > 0 \\ (\xi + \xi_1), & h = 0 \\ \frac{1}{h} \sin[h(\xi + \xi_1)], & h < 0 \end{cases} \quad (53)$$

where h and ξ_1 are integration constants.

Taking into account (51) and (52), we find from the harmonic coordinate condition (3), the following expression of the metric function $\alpha(\xi)$:

$$\alpha(\xi) = \frac{M}{2} \left(\frac{3}{2} + \frac{2}{A}\right) \ln \frac{M}{AT^2(h, \xi + \xi_1)} \quad (54)$$

It then follows that:

$$\beta(\xi) = \left(\frac{2+A}{4+3A}\right) \alpha(\xi); \quad \gamma(\xi) = \left(\frac{A}{4+3A}\right) \alpha(\xi) \quad (55)$$

Introducing the results (49) and (55) into (5), the component $\binom{1}{1}$ of Einstein tensor G_μ^ν becomes:

$$(\alpha')^2 = \frac{(4+3A)^2}{3A^2+8A+4} e^{2\alpha} \left[e^{\frac{-4-2A}{4+3A}\alpha} - k(mS - H(I_T)) \right] \quad (56)$$

hence

$$\int \frac{d\alpha}{\frac{(4+3A)}{\sqrt{3A^2+8A+4}} e^\alpha \sqrt{\left[e^{\frac{-4-2A}{4+3A}\alpha} - k(mS - H(I_T)) \right]}} = \pm (\xi + \xi_0), \quad \xi_0 = const \quad (57)$$

Using the expression (37), we can pass from the metric function $\alpha(\xi)$ to the invariant function $I_T(\xi)$:

$$e^{\alpha(\xi)} = \sqrt{\frac{b_1}{I_T}}, \quad d\alpha = -\frac{1}{2} \frac{dI_T}{I_T} \quad (58)$$

Inserting (58) into (57), one gets the general following solutions:

$$\int \frac{dI_T}{\sqrt{I_T} \sqrt{\left\{ \left(\frac{I_T}{b_1}\right)^{\frac{2+A}{4+3A}} - \kappa[(mS - H(I_T))]\right\}}} = \pm 2 \frac{(4+3A)\sqrt{b_1}}{\sqrt{3A^2+8A+4}} (\xi + \xi_0) \quad (59)$$

The general solutions (59) depend on the arbitrary function $H(I_T)$. Thus, setting a concrete form of the function $H(I_T)$, from (59), we can determine the invariant function $I_T(\xi)$. Then, knowing $I_T(\xi)$, we can find the metric function $\alpha(\xi)$ from (37), the functions $\beta(\xi)$ and $\gamma(\xi)$ from (55). Let us remark that at this step, we can clearly obtain $G(S, P)$ and $D(S, P)$ as functions of the metric function $\alpha(\xi)$, using the relation (46) or (48).

Note that from the relation $I_T = b_1 e^{-2\alpha(\xi)}$, we can study the regular properties of the obtained solutions. We can verify the localisation of the energy density and the energy per unit invariant volume $T_0^0 \sqrt{3-g}$. Finally, we can find the total energy of the spinor field and establish the localisation properties of the solutions. In the sequel, we analyze the influence of the nonlinearity of the spinor field of the general solutions obtained previously for concrete nonlinear terms.

Discussion:-

The principle aim, in this section, is to show the importance of the nonlinearity of the spinor field in order to obtain the regular solutions with bounded energy density and finite total energy (Soliton model). To this end, we choose the nonlinear terms in the spinor field density lagrangian under the following polynomial form:

$$H(I_T) = \lambda I_T^n, \quad n > 1 \quad (60)$$

with λ the parameter of nonlinearity and n the power of nonlinearity. Let us emphasize that we consider and analyze separately the four cases: $H(I_T) = 0, \lambda \neq 0$ and $n = 1, \lambda \neq 0$ and $n > 1, \lambda = -\omega_0^2 < 0$ and $n > 1$.

- For $H(I_T) = 0$, from (19), we obtain Dirac's linear equation as follows:

$$ie^{-\alpha} \gamma^{-1} \left(\partial_\xi + \frac{1}{2} \alpha' \right) \psi + \frac{i}{2} \bar{\gamma}^2 e^{-\beta} \psi \cot \theta - m \psi = 0 \quad (61)$$

According to (59), by assuming that $\frac{2+A}{4+3A} = 1$, we get ;

$$I_T(\xi) = b_1 \exp \left[\frac{(8+6A)(\xi + \xi_0)}{\sqrt{3A^2 + 8A + 4}} \right]. \quad (62)$$

From $I_T = b_1 e^{-2\alpha(\xi)}$, we deduce :

$$\alpha(\xi) = \beta(\xi) = -\frac{(8+6A)}{\sqrt{3A^2 + 8A + 4}} (\xi + \xi_0) \quad (63)$$

$$\gamma(\xi) = -\frac{A}{\sqrt{3A^2 + 8A + 4}} (\xi + \xi_0) \quad (64)$$

According to (12),

$$T_0^0 = 0 \quad (65)$$

The energy density is not localized in the case under consideration.

Let us note that in the linear case soliton-like solutions are absent. Therefore, the nonlinear terms in the nonlinear field equations are very important in the formation of regular localized soliton-like solutions. Moreover, it is necessary to introduce nonlinear terms that characterize the field interactions in the lagrangian. The similar results are obtained in [4]. In the sequel, we deal with Heisenberg-Ivanenko type nonlinear spinor field equation.

- For $\lambda \neq 0$ and $n = 1$, we have Heisenberg-Ivanenko type nonlinear spinor field equation as defined in [2] under the following expression :

$$ie^{-\alpha} \gamma^{-1} \left(\partial_\xi + \frac{1}{2} \alpha' \right) \psi + \frac{i}{2} \bar{\gamma}^2 e^{-\beta} \psi \cot \theta - m \psi + 4\lambda (\bar{\psi} \psi) \psi = 0 \quad (66)$$

Let us remark that the expression obtained for the invariant function $I_T(\xi)$ is similar to the relation obtained in the linear case, only now the constant $\frac{(8+6A)}{\sqrt{3A^2 + 8A + 4}}$ is replaced by the constant $\frac{(8+6A)\sqrt{1+k\lambda b_1}}{\sqrt{3A^2 + 8A + 4}}$. In these conditions, we get:

$$I_T(\xi) = b_1 \exp \left[\frac{(8+6A)\sqrt{1+k\lambda b_1}}{\sqrt{3A^2 + 8A + 4}} (\xi + \xi_0) \right]. \quad (67)$$

As for the metric functions, they are defined under the forms:

$$\alpha(\xi) = \beta(\xi) = -\left[\frac{(4+3A)\sqrt{1+k\lambda b_1}}{\sqrt{3A^2 + 8A + 4}} (\xi + \xi_0) \right], \quad (68)$$

$$\gamma(\xi) = -\left[\frac{A\sqrt{1+k\lambda b_1}}{\sqrt{3A^2 + 8A + 4}} (\xi + \xi_0) \right]. \quad (69)$$

Our results prove that the metric functions $g_{00} = e^{2\gamma(\xi)}, g_{11} = -e^{2\alpha(\xi)}, g_{22} = -e^{2\beta(\xi)}, g_{33} = -e^{2\beta(\xi)} \sin \theta$ and the invariant function $I_T(\xi) = b_1 e^{-2\alpha(\xi)}$ are regular and bounded.

Substituting $H(I_T) = \lambda I_T$ into (12), the energy density is given by:

$$T_0^0(\xi) = \lambda b_1 \exp \left[\frac{(8+6A)\sqrt{1+k\lambda b_1}}{\sqrt{3A^2 + 8A + 4}} (\xi + \xi_0) \right]. \quad (70)$$

According to (66), the distribution of the spinor field energy density per unit invariant volume is

$$\Omega(\xi) = \lambda b_1 \exp \left[\frac{A\sqrt{1+k\lambda b_1}}{\sqrt{3A^2 + 8A + 4}} (\xi + \xi_0) \right] \sin \theta. \quad (71)$$

The expression of the energy density T_0^0 defined by (70) is localized. From (71) the energy density per unit invariant volume is localized too when $\xi \in [0, \xi_c]$. Moreover, the total energy $E = \int_0^{\xi_c} \Omega(\xi) d\xi$ is finite. Its finite quantity is defined as follows $E = \frac{\lambda b_1 \sin \theta}{A} \sqrt{\frac{3A^2 + 8A + 4}{1 + \kappa \lambda b_1}} \exp \left[\frac{A \xi_c \sqrt{1 + \kappa \lambda b_1}}{\sqrt{3A^2 + 8A + 4}} - 1 \right]$, for ξ_0 considered as null.

Let us remark that the obtained solution describes a nonlinear spinor field configuration with regular localized energy density $T_0^0(\xi)$, finite total energy E and regular metric functions. Thus, the equation (66) possesses a soliton-like solution. In the sequel, we deal with the generalization corresponding to $\lambda \neq 0$ and $n > 1$.

- For $\lambda \neq 0$ and $n > 1$, we have

$$I_T(\xi) = \left\{ \frac{1}{\sqrt{k \lambda b_1 \sinh \left[\frac{(4+3A)}{\sqrt{3A^2 + 8A + 4}} (n-1)(\xi + \xi_0) \right]}} \right\}^{\frac{2}{n-1}} \quad (72)$$

With the condition $b_1 = a_1^2 > 0$, we deduce from (37) and (72) the expression of the metric function $\alpha(\xi)$ as follows:

$$\alpha(\xi) = \ln \left\{ \frac{a_1}{\left[\sqrt{k \lambda a_1^2 \sinh \left[\frac{(4+3A)}{\sqrt{3A^2 + 8A + 4}} (n-1)(\xi + \xi_0) \right]} \right]^{\frac{1}{n-1}}} \right\} \quad (73)$$

Taking into account (55) and (73), we get:

$$\beta(\xi) = \ln \left\{ \frac{a_1}{\left[\sqrt{k \lambda a_1^2 \sinh \left[\frac{(4+3A)}{\sqrt{3A^2 + 8A + 4}} (n-1)(\xi + \xi_0) \right]} \right]^{\frac{1}{n-1}}} \right\} \quad (74)$$

$$\gamma(\xi) = \left(\frac{A}{4+3A} \right) \ln \left\{ \frac{a_1}{\left[\sqrt{k \lambda a_1^2 \sinh \left[\frac{(4+3A)}{\sqrt{3A^2 + 8A + 4}} (n-1)(\xi + \xi_0) \right]} \right]^{\frac{1}{n-1}}} \right\} \quad (75)$$

The energy density is defined by:

$$T_0^0(\xi) = \lambda(2n - 1) \left\{ \frac{1}{\sqrt{a_1^2 \lambda k \sinh \left[\frac{(4+3A)}{\sqrt{3A^2 + 8A + 4}} (n-1)(\xi + \xi_0) \right]}} \right\}^{\frac{2n}{n-1}} \quad (76)$$

Let us establish the expression of the energy density per unit invariant volume. For this purpose, using (76), we find

$$f(\xi) = \lambda(2n - 1) \left\{ \frac{1}{\sqrt{a_1^2 \lambda k \sinh \left[\frac{(4+3A)}{\sqrt{3A^2 + 8A + 4}} (n-1)(\xi + \xi_0) \right]}} \right\}^{\frac{2n}{n-1}} \exp[v(\xi)] \sin \theta. \quad (77)$$

where

$$v(\xi) = \left(\frac{8+5A}{4+3A} \right) \ln \left\{ \frac{a_1}{\left[\sqrt{k \lambda a_1^2 \sinh \left[\frac{(4+3A)}{\sqrt{3A^2 + 8A + 4}} (n-1)(\xi + \xi_0) \right]} \right]^{\frac{1}{n-1}}} \right\} \quad (78)$$

The soliton-like solutions do not exist when the nonlinear term is chosen under the form $H(I_T) = \lambda I_T^n$, $\lambda \neq 0$ and $n > 1$. Indeed the invariant function $I_T(\xi) \rightarrow \infty$ and the energy density $T_0^0(\xi) \rightarrow \infty$ when $\xi \rightarrow 0$. Nevertheless, the metric functions are regular and the metric is stationary. Our results carried out that this form is not appropriated in order to obtain soliton model.

- For $\lambda = -\omega_0^2 < 0$, from (59), we obtain the expression of the invariant function as follows:

$$I_T(\xi) = \left\{ \frac{1}{\sqrt{k\omega_0^2 a_1^2 \cos \hbar \left[\frac{(4+3A)}{\sqrt{3A^2+8A+4}}(n-1)(\xi+\xi_0) \right]}} \right\}^{\frac{2}{n-1}} \quad (79)$$

From the expression (79), we note that the invariant function $I_T(\xi)$ is continuous and bounded when $\xi \in [0, \xi_c]$. Considering the expressions (37), (55) and (79), we deduce the expressions of the metric functions as follows:

$$\alpha(\xi) = \ln \left\{ \frac{a_1}{\left[\sqrt{k\omega_0^2 a_1^2 \cos \hbar \left[\frac{(4+3A)}{\sqrt{3A^2+8A+4}}(n-1)(\xi+\xi_0) \right]} \right]^{\frac{1}{n-1}}} \right\} \quad (80)$$

Taking into account (55) and (8), we get:

$$\beta(\xi) = \ln \left\{ \frac{a_1}{\left[\sqrt{k\omega_0^2 a_1^2 \cos \hbar \left[\frac{(4+3A)}{\sqrt{3A^2+8A+4}}(n-1)(\xi+\xi_0) \right]} \right]^{\frac{1}{n-1}}} \right\} \quad (81)$$

$$\gamma(\xi) = \left(\frac{A}{4+3A} \right) \ln \left\{ \frac{a_1}{\left[\sqrt{k\omega_0^2 a_1^2 \cos \hbar \left[\frac{(4+3A)}{\sqrt{3A^2+8A+4}}(n-1)(\xi+\xi_0) \right]} \right]^{\frac{1}{n-1}}} \right\} \quad (82)$$

As for the energy density of the spinor field, we have

$$T_0^0(\xi) = -\omega_0^2(2n-1) \left\{ \frac{1}{\sqrt{k\omega_0^2 a_1^2 \cos \hbar \left[\frac{(4+3A)}{\sqrt{3A^2+8A+4}}(n-1)(\xi+\xi_0) \right]}} \right\}^{\frac{2n}{n-1}} \quad (83)$$

The energy density per unit invariant volume is

$$f(\xi) = -\omega_0^2(2n-1) \left\{ \frac{1}{\sqrt{k\omega_0^2 a_1^2 \cos \hbar \left[\frac{(4+3A)}{\sqrt{3A^2+8A+4}}(n-1)(\xi+\xi_0) \right]}} \right\}^{\frac{2n}{n-1}} \exp[v(\xi)] \sin \theta \quad (84)$$

Since the function $I_T(\xi)$ is continuous and bounded when $\xi \in [0, \xi_c]$, hence the energy density $T_0^0(\xi)$ and the energy density per unit invariant volume $T_0^0(\xi)\sqrt{3-g}$ are limited. In this case, the total energy $E = \int_0^{\xi_c} f(\xi) d\xi$ is finite. Therefore, these solutions are soliton-like solutions and can be used in order to describe the configuration of elementary particles. Let us define the functions $\Lambda(\xi)$, $S(\xi)$, $P(\xi)$, $D(S,P)$ and $G(S,P)$ as functions of $e^{\alpha(\xi)}$. By doing so, from (48), we have:

$$D(S,P) = \frac{2ne^{-\alpha(\xi)}}{ka_1 \cos \hbar [a_1 \sqrt{2} \Lambda(\xi)] \cos \hbar^2 \left[\frac{(4+3A)}{\sqrt{3A^2+8A+4}}(n-1)(\xi+\xi_0) \right]} \quad (85)$$

$$G(S,D) = - \frac{2n \tan \hbar [a_1 \sqrt{2} \Lambda(\xi)] e^{-\alpha(\xi)}}{ka_1 \cos \hbar^2 \left[\frac{(4+3A)}{\sqrt{3A^2+8A+4}}(n-1)(\xi+\xi_0) \right]} \quad (86)$$

As for the functions, $S(\xi)$ and $P(\xi)$, their expressions obtained in (46) and (47) are valid. By virtue of (41) and (79), we have:

$$\Lambda(\xi) = - \frac{4n\sqrt{3A^2+8A+4}}{a_1^2 k(n-1)(4+3A)} \tan \hbar \left[\frac{(4+3A)}{\sqrt{3A^2+8A+4}}(n-1)(\xi+\xi_0) \right] + b_2 \quad (87)$$

The set equations (22)-(25) has the following solutions:

$$V_1(\xi) = \frac{1}{4\sqrt{b_1}} [A_1 e^{k_1(\xi)\Lambda(\xi)} + A_2 e^{k_2(\xi)\Lambda(\xi)} + A_3 e^{k_3(\xi)\Lambda(\xi)} + A_4 e^{k_4(\xi)\Lambda(\xi)}] e^{-\frac{1}{2}\alpha(\xi)}, \quad (88)$$

$$V_1(\xi) = \frac{1}{4\sqrt{b_1}} [A_5 e^{k_5(\xi)\Lambda(\xi)} + A_6 e^{k_6(\xi)\Lambda(\xi)} + A_7 e^{k_7(\xi)\Lambda(\xi)} + A_8 e^{k_8(\xi)\Lambda(\xi)}] e^{-\frac{1}{2}\alpha(\xi)}, \quad (89)$$

$$V_1(\xi) = \frac{1}{4\sqrt{b_1}} [A_1 e^{k_1(\xi)\Lambda(\xi)} + A_3 e^{k_3(\xi)\Lambda(\xi)} - A_2 e^{k_2(\xi)\Lambda(\xi)} - A_4 e^{k_4(\xi)\Lambda(\xi)}] e^{-\frac{1}{2}\alpha(\xi)}, \quad (90)$$

$$V_1(\xi) = \frac{1}{4\sqrt{b_1}} [A_5 e^{k_5(\xi)\Lambda(\xi)} + A_7 e^{k_7(\xi)\Lambda(\xi)} - A_6 e^{k_6(\xi)\Lambda(\xi)} - A_8 e^{k_8(\xi)\Lambda(\xi)}] e^{-\frac{1}{2}\alpha(\xi)}, \quad (91)$$

where $A_1, A_2, A_3, A_4, A_5, A_6, A_7$ and A_8 are integration constants. For details about the determination of the analytic forms of the functions $V_\delta(\xi)$, refer to [7] and references therein.

Let us find the charge density and the chronometric invariant spin tensor. Using solutions (88)-(91), as we consider the field configuration to be a static one, the spatial components j^1, j^2 and j^3 of the spinor current $j^\mu = \bar{\psi}\gamma^\mu\psi$ vanish. The non-null component j^0 is defined as follows:

$$j^0 = \bar{F}(\Lambda(\xi)) \cdot e^{-\alpha-\gamma}. \quad (92)$$

The component j^0 leads to the determination of the charge density of spinor field that has the following chronometric-invariant form:

$$\rho(\xi) = (j_0 j^0)^{\frac{1}{2}} = \bar{F}(\Lambda(\xi)) \cdot e^{-\alpha} \quad (93)$$

The total charge of the spinor field is:

$$Q = \int_0^{\xi_c} \rho(\xi) \sqrt{3-g} d\xi. \quad (94)$$

Let us consider the spin tensor. As defined in [10], we have:

$$S^{\mu\nu,\varepsilon} = \frac{1}{4} \bar{\psi} \{ \gamma^\varepsilon \sigma^{\mu\nu} + \sigma^{\mu\nu} \gamma^\varepsilon \} \psi \quad (95)$$

Taking into account the expression (95), we write explicitly the component $S^{ik,0}$ ($i, k = 1, 2, 3$) of the spatial density of spin vector as follows:

$$S^{ik,0} = \frac{1}{4} \bar{\psi} \{ \gamma^0 \sigma^{ik} + \sigma^{ik} \gamma^0 \} \psi = \frac{1}{2} \bar{\psi} \gamma^0 \sigma^{ik} \psi \quad (96)$$

The components of the tensor $S^{ik,0}$ are given by:

$$S^{12,0} = 0, S^{13,0} = 0, S^{23,0} = \frac{1}{2} \bar{V}_\delta \gamma^0 \sigma^{23} V_\delta e^{-\alpha} \quad (97)$$

Let us define the chronometric invariant spin tensor. To this end, using the equality (97), we have:

$$S_{\text{chl}}^{23,0} = (S_{23,0} \cdot S^{23,0})^{\frac{1}{2}} = \bar{F}(\Lambda(\xi)) \cdot e^{-\alpha} \quad (98)$$

Finally, we obtain the projection of spin vector S_1 on ξ axis.

$$S_1 = \int_0^{\xi_c} \bar{F}(\Lambda(\xi)) \cdot e^{-\alpha} \sqrt{3-g} d\xi \quad (99)$$

From (93), we conclude that the charge density is localized when $\xi \in [0, \xi_c]$ as well as the charge density unit invariant volume $\rho(\xi) \sqrt{3-g}$. Thus, the total charge of the spinor field is bounded. Then, according to (93) and (98), the expression for chronometric invariant tensor of spin $S_{\text{chl}}^{23,0}$ coincides with the charge density $\rho(\xi)$ one. Therefore, the conclusion made for $\rho(\xi)$ and Q will be valid for $S_{\text{chl}}^{23,0}$ and S_1 . Moreover, $S_{\text{chl}}^{23,0}$ is localized and S_1 is limited.

Concluding Remarks:-

In the present research work, we obtained the exact static spherical symmetric solutions to the nonlinear spinor field and Einstein's equations. We investigated in detail equations with power and polynomial nonlinearities. The obtained solutions are regular with localized energy density and finite total energy. In addition, the total charge Q and the total spin S_1 are also finite. The forthcoming paper will deal with interaction spinor and scalar fields equations in gravitational theory.

Conflicts of Interest:-

The authors declare no conflicts of interest regarding the publication of this paper.

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