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*RESEARCH ARTICLE*

## FINITE VOLUME METHODDISCRETIZATIONOF MODIFIEDNAVIER-STOKES EQUATION

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***Abstract***

This study has come up with a numerical scheme that arises from finite volume discretizationof Modified Navier-Stokes Equation.Modified Navier-Stokes Equation in the x-z axis was coupled with continuity equation to obtain the Pressure Equation. Using the pressure equation, pressure field can bedetermined in each control volume on a staggered grid.



## Introduction:-

Navier-Stokes Equation is a partial differential equation representing a force of fluid flow field.Frenchman M. Navier and Englishman G.Stokes independently obtained the equation in firsthalf of the nineteenth century and its named to their honour.It can describe many phenomenaof scientific interest. Despite its wide application, it still has challenges in its existence of smooth solutions becauseof its non-linearity terms and lack of explicit equation for pressure.

This study formulated a numerical solution for a Modified Navier-Stokes Equation in the x and z directions of the Cartesian plane using Finite Volume Method (FVM).

**Modified Navier-Stokes Equation**

Three-dimensional Navier-Stokes Equation is given in the form:

$$[\frac{\partial}{\partial t}(\rho\vec{v})+\nabla.(\rho\vec{v}\vec{v})=-\nabla P+\nabla.(\bar{\tau})+\rho f] \qquad (1)$$

The RHS (right hand side) of Navier-Stokes Equation are the sum of forces acting on a flowing fluid element.The force due to viscosity, shear stress is defined by the gradient:

$$\vec{F}_{vs}=\nabla.\tau dV=\nabla.\tau dxdydz \qquad (2)$$

where $\tau$ is the stress tensor.

The viscous term in the Navier-Stokes Equation is modified into a diffusion term given by:

$$\vec{F}=\nabla.(\Gamma\nabla\tau_{\omega})$$

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where $\Gamma$ is a diffusion coefficient and wall shear stress $(\tau_w)$
If equation(3) replaces the viscous term in Navier-Stokes Equation, it changes to Modified Navier-Stokes Equation. Modified Navier-Stokes Equation models fluid flow on the boundary whichfluid flows.This conceptof boundary layer was first introduced by a German scientist LudwingPrandtl in the year 1904.Modified Navier-Stokes Equation in x and z directions coupled withContinuity Equation in a Cartesian plane are given as follows:

x- direction

$$\frac{\partial \rho u}{\partial t}+\nabla.(\rho u\vec{v})=-\frac{\partial P}{\partial x}+\nabla.(\Gamma\nabla\tau_{\omega}) \quad (4)$$

z- direction

$$\frac{\partial \rho w}{\partial t}+\nabla.(\rho w\vec{v})=-\frac{\partial P}{\partial z}+\nabla.(\Gamma\nabla\tau_{\omega})+\rho g_z \quad (5)$$

Continuity Equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad (6)$$

Continuity Equation expresses incompressibility of the fluid that is flowing.
$u$ - is velocity component of the fluid in the x-direction, $w$ - is velocity component of the fluid in the z-direction, $\rho$ - density of the fluid flowing, $\tau_\omega$ - wall shear stress, $\Gamma$ - wall shear stress diffusion coefficient.

**3.0 Discretizationof Modified Navier-Stoke Equation Using Finite Volume Methodon a Staggered Grid**

Finite Volume Method exhibits the following features:

(i) Formulations can be based surface integral of normals that guaranteeconservation properties throughout thedomain.

(ii) Complex geometries and unstructured meshes are easily accommodated, no co-ordinate transformation is needed.

(iii) Neumann boundary conditions are easily enforced.

Finite volume method is preferred over other numerical methods because it ensures that discretization is conservative locally and globally, i.e. mass, momentum and energy areconserved in discrete sense.
In staggered grid discretization the scalars are stored at the centre of the control volume and velocity are centered at the faces of control volumes. If a uniform grid is used, the velocity locations are exactly at the midway between the grid points. Staggered grid for dependent variables was first used by Harlow and Welch intheir well-knownMac (Makerand Cell) method. Since then it has been used by many researchers specificallySIMPLE (Semi-Implicit Method for Pressure Linked Equations) Procedure by of Patanker and Spalding (1972).

Integration of equations(4),(5)with respect to time dependent control volume v (t) is:

$$\int_{v(t)}\left[\frac{\partial \rho u}{\partial t}+\nabla.(\rho u\vec{v})=-\frac{\partial P}{\partial x}+\nabla.(\Gamma\nabla\tau_{\omega})\right]dv(t) \quad (7)$$

$$\int_{v(t)}\left[\frac{\partial \rho w}{\partial t}+\nabla.(\rho w\vec{v})=-\frac{\partial P}{\partial z}+\nabla.(\Gamma\nabla\tau_{\omega})+\rho g_z\right]dV(t) \quad (8)$$

Using divergence theorem, the volume integrals are converted into surface integrals.
Finite Volume Method discretized Modified Navier-Stokes Equation in the x-direction is given by:

$$\left[(\rho u)_P^{n+1}-(\rho u)_P^{n}\right]\Delta x\Delta z+(\rho u)_P^{n+1}(\Delta z)_e-(\rho u)_W^{n+1}(\Delta z)_w+(\rho w)_P^{n+1}(\Delta x)_n-(\rho w)_S^{n+1}(\Delta x)_s=D_P(\tau_\omega)_P^{n+1}+$$

$$\sum_{nb}D_{nb}(\tau_\omega)_{nb}+(P_P-P_E)^{n+1}\Delta z \quad (9)$$

$$(\rho u)^{n}\Delta x\Delta z-(\rho u)_{P}^{n+1}(\Delta z)_{e}+(\rho u)_{W}^{n+1}(\Delta z)_{w}-(\rho w)_{P}^{n+1}(\Delta x)_{n}+(\rho w)_{S}^{n+1}(\Delta x)_{s}+D_{P}(\tau_{\omega})_{P}^{n+1}=b \quad (10)$$

where b is the mass source term. Hence equation (9) can be written as:

$$(\rho u)_{P}^{n+1}\Delta x\Delta z=\sum_{nb}D_{nb}(\tau_{nb})+b+(P_{P}-P_{E})^{n+1}\Delta z$$

$$a\ u_{e}=\sum_{nb}D_{nb}^{u}(\tau_{\omega})_{nb}+b^{u}+(P_{P}-P_{E})^{n+1}\Delta z$$

$$u_{e}=\frac{\sum_{nb}D_{nb}^{u}(\tau_{\omega})_{nb}+b^{u}}{a}+\frac{\Delta z}{a}(P_{P}-P_{E})$$

$$u_{e}=\frac{\sum_{nb}D_{nb}^{u}(\tau_{\omega})_{nb}+b^{u}}{a}+d_{e}(P_{P}-P_{E})^{n+1}$$

$$u_{e}=\tilde{u}+d_{e}(P_{p}-P_{E})^{n+1}\ \text{where}\ d_{e}=\frac{\Delta z}{a},\ \tilde{u}=\frac{\sum_{nb}D_{nb}^{u}(\tau_{\omega})_{nb}+b^{u}}{a} \quad (11)$$

$u_e$ gives the velocity in the x-direction at (n+1) control volume face value.

Finite Volume Methoddiscretized Modified Navier-Stokes Equation in the z-direction is given by:

$$\left[(\rho w)^{n+1}-(\rho w)^{n}\right]\Delta x\Delta z+(\rho w)_{P}^{n+1}(\Delta x)_{n}-(\rho w)_{S}^{n+1}(\Delta x)_{s}+(\rho w)_{P}^{n+1}(\Delta z)_{e}-(\rho w)_{W}^{n+1}(\Delta z)_{w}=(P_{P}-P_{N})^{n+1}(\Delta x)$$

$$+D_{P}(\tau_{\omega})_{P}^{n+1}+\sum_{nb}D_{nb}(\tau_{\omega})_{nb}+(\rho g_{z})^{n+1}\Delta x\Delta z$$

where mass term can be written as:

$$(\rho w)^{n}\Delta x\Delta z+(\rho w)_{S}^{n+1}(\Delta x)_{s}-(\rho w)_{P}^{n+1}(\Delta x)_{n}+(\rho w)_{W}^{n+1}(\Delta z)_{w}-(\rho w)_{P}^{n+1}(\Delta z)_{e}+D_{P}(\tau_{\omega})_{P}^{n+1}+(\rho g_{z})^{n+1}\Delta x\Delta z=b$$

(12)

Hence the equation (12) can be rewritten as:

$$w_{n}=\frac{\sum_{nb}D_{nb}(\tau_{\omega})_{nb}^{w}+b}{a}+\frac{\Delta x}{a}(P_{p}-P_{N})^{n+1} \quad (13)$$

$$w_{n}=\tilde{w}_{n}+d_{n}(P_{p}-P_{N})^{n+1}$$

where: $d_{n}=\frac{\Delta x}{a},\ \tilde{w}_{n}=\frac{\sum_{nb}D_{nb}(\tau_{\omega})_{nb}^{w}+b^{w}}{a}$

$w_n$ is the velocity in the z-direction at (n+1) control volume face value

**Discretization ofContinuity Equation**

In two dimension x-z axis is given by differential equation:

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \quad (14)$$

Integration of equation (14)over control volume (CV) gives:

$$\int_{cv}\left[\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right]dv$$

$$(u_{i+1}-u_i)\Delta z = -(w_{j+1}-w_j)\Delta x \qquad (15)$$

which is Finite Volume Method discretized form of Continuity Equation where $u_{i+1}, u_i$ are velocities in the x-direction and $w_{j+1}, w_j$ are velocities in the z-direction.

**Pressure Equation**

Substituting equation (11) and equation (13) into equation (15) gives:

$$\left[\tilde{u}_n + d_e(P_P - P_E) - u_i\right]\Delta z = -\left[\tilde{w}_n + d_n(P_P - P_N) - w_i\right]\Delta x$$

$$\tilde{u}_n\Delta z + d_e\Delta z P_P - d_e\Delta z P_E - u_i\Delta z = -\tilde{w}_n\Delta x - d_n\Delta x P_P + d_n\Delta x P_N + w_i\Delta x$$

$$(d_e\Delta z + d_n\Delta x)P_p - d_n\Delta x P_N - d_e\Delta z P_E + \tilde{u}_i\Delta z + \tilde{w}_i\Delta x - w_i\Delta x - u_i\Delta z = 0$$

$$a_P P_P - a_n P_N - a_e P_E + (\tilde{u}_i - u_i)\Delta z + (\tilde{w}_i - w_i)\Delta x = 0 \qquad (16)$$

where $a_n = d_n\Delta x, a_e = d_e\Delta z, a_p = d_e\Delta z + d_n\Delta x$

Substitution of $\tilde{u}_i = \sum_{nb} D^u_{nb}(\tau_\omega)^u_{nb} + b^u$ and $\tilde{w}_i = \sum_{nb} D^w_{nb}(\tau_\omega)^w_{nb} + b^w$ into equation (15) and after algebraicsimplification gives:

$$a_p P_p = a_e P_E + a_n P_N + b^p \qquad (17)$$

where $b^p = -\Delta z \sum_{nb} D^u_{nb}(\tau_\omega)^u_{nb} - \Delta z b^u + u_i\Delta z + w_i\Delta x - \Delta x \sum_{nb} D^w_{nb}(\tau_\omega)^w_{nb} - \Delta x b^w$

which is the Pressure Equation.

## Conclusion:-

The Pressure Equation is used to estimate the values pressure P(ij) on a staggered grid in the x-z directions oneach control volume.

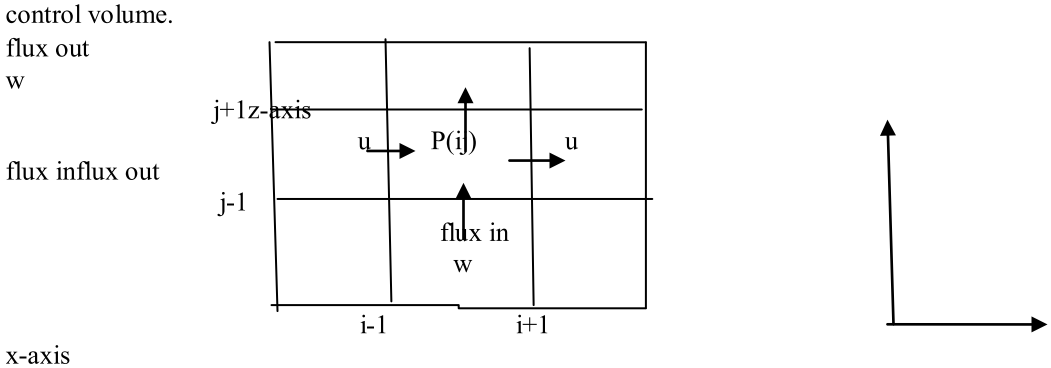


Fig 1: 3x3 mesh for a staggered grid

Thereafter horizontal velocity (u) and vertical velocity (w)at each control volume interfaces can be approximated using SIMPLER (Semi Implicit Method for Pressure Linked Equations Revised) algorithm.

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