Journal Homepage: -www.journalijar.com

INTERNATIONAL JOURNAL OF ADVANCED RESEARCH (IJAR)

Article DOI:10.21474/IJAR01/20742
DOI URL: http://dx.doi.org/10.21474/IJAR01/20742

RESEARCH ARTICLE

THE CONTINUOUS AND IMPULSIVE GRAZING MODEL IN A SAHELIAN REGION

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Manuscript Info

Manuscript History
Received: 15 Febuary 2025
Final Accepted: 18 March 2025
Published:April 2025

Key words:-
Pastoralism; Grazing; continuous harvest ing; Impulsive harvesting; continuation theorem.

Abstract

Pastoralism in the Sahel plays a major role in the economy of West African countries. With a wide diversity of livestock species, it contributes to the food and nutritional security of rural and urban households. So, for reasons of survival, the shepherd can harvest from the herd population. Thus, in this paper, we build and analyze a hybrid mathematical model of resource-herd interactions in a Sahelian region by first taking into account a continuous harvest for various needs. As several months, or even years, can elapse between two harvests, it is more realistic to consider that this harvesting takes place impulsively, as it very often occurs at specific times for known reasons (sale, bequest, slaughter, etc.). We have carried out a qualitative analysis of our model, and numerical simulations are provided to illustrate our theoretical results and support the discussion. theoretical results and support the discussion.

Introduction:-

In many Sahelian countries, pastoralism is both a production activity and a way of life that can be understood as an occupation stemming from a genuine vocation (Baxter, 1994)[16]. Numerous attempts to define the pastoral economy have been made, drawing on Swift's (1988) contribution. She defines pastoral production systems as "those in which at least 50% of gross household income (i.e. the value of marketed production and the estimated value of subsistence production consumed by households) is derived from livestock or livestock-related activities (e.g. caravan trade), or wheremore than 15% of household food energy consumption consists of milk or dairy products produced by the household"[16]. One of the main objectives of economic research on pastoral livestock systems is to examine the viability of these systems and the possibilities for sustainably to improve livelihoods. The adoption of research-generated interventions on pastoral livestock funded he the f a useful starting point for understanding how and why pastoralists sell livestock. Pastoralists logically associate wealth with the accumulation of livestock numbers rather than money. This strategy is based on the high yields of livestock relative to cash, a natural resource base that supports livestock production, and the limited financial services available in pastoral areas.In West African countries, in terms of contributions to national and regional economies, livestock accounts for an average of 44% of gross domestic product (GDP), and it is reasonable to assume that most of this economic

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activity is derived from pastoral systems[17].Government systems for collecting data on the formal livestock trade in pastoral areas are often poor, and there is also a very substantial informal trade in livestock that is difficult to measure, including cross-border trade in places where government presence is limited. Generally speaking, this informal trade is not taken into account in formal statistics on national livestock economies.

Household and herd growth strategies in livestock marketing suggest that herders tend to sell their livestock only when they have immediate cash needs, e.g. to buy food or medicine, or for ceremonies, or to pay school fees. This behavior minimizes herd reduction and thus contributes to herd and financial growth. As cash requirements are often seasonal, so are livestock sales. As a result, there is no regular supply of livestock to markets on a monthly basis, but rather a seasonal one. Another consequence of indigenous herd growth strategies is that pastoralists tend not to be as reactive as one might expect to rising livestock prices on the markets[17]. Not only do they mainly sell when they need to sell, but owning sheep or goats as part of a mixed herd is often a convenient sales unit; they don't necessarily need to sell larger, more valuable livestock species to meet their cash needs. When there is an accessible market, livestock prices rise and staple food prices remain constant, a poor pastoral household can sell fewer animals to cover household food needs. In other words, higher livestock prices may result in fewer animals being sold, not more. The basic economics of pastoral households and the resulting marketing behavior explain why livestock marketing in pastoral areas differs greatly according to wealth level. As described in USAID's brief Introduction, the main livestock suppliers to local and international markets are relatively well-off households[17].

The study and design of bio-economic models is attracting a great deal of interest with regard to biodiversity and the economics of long-term gains. Researchers are striving to produce potentially beneficial results to ensure sustainable ecosystem growth and also to maintain sustainable prosperity. Thus, more recently, the study of population dynamics with harvesting has become an interesting topic in bio-economics because of its importance linked to the optimal management of renewable resources [1].

Resource-herbivore interaction models can be seen as a special case of prey-predator models. Let's consider R as the available resource and we assume that, in the absence of herbivores, the resource grows logistically:

$$\frac{dR(t)}{dt} = r\left(1 - \frac{R(t)}{K}\right)R(t),$$

with K denoting the carrying capacity of livestock and r the intrinsic production of forage resource. The intrinsic growth rate r of resources can be written as $r = r_0 - r_1$ with. $r_0 = r_{01}p + r_{02}$ as given in [1]and [2], p denotes precipitation, r_{02} is the growth rate of grasses and the like without so precipitation and r_1 is the rate of disappearance due to many reasons (competition, respiration and human harvesting) different from herd feeding. The particularity of this model, compared with classic prey-predator models in the literature, is that the intrinsic production rate can be negative ($r < 0$). This situation is frequently observed in the Sahelian region, where rainfall is often scarce. A worsening of this situation can lead to transhumance.

In the study of herbivore-resource interactions, it is important to determine which response functions describe the quantity of resources consumed by a herd animal per unit of time. We consider here that herbivores feed mainly on these resources with saturation of the holling II type, defined by:

$$\phi(R) = \frac{aR}{b+R},$$

where a is the herbivore consumption rate per unit of time and b is the time it takes for the herbivores to recover the resources.

Nowadays, in addition to effet and climatic events, speculation, transmission of price rises, price volatility, effets of substitution of imported products by local production...exacerbate the instability of the terms of trade (David- Benz et al, 2010). At this level, therefore, we need to take into account the harvesting of individuals from the herd for survival needs, and therefore not necessarily taking into account the number of LUs in the herd. The model to be studied initially takes into account the continuous harvesting of individuals from the herd. We consider the following herbivore harvesting function:

$$\omega(H) = \frac{hH}{c+H},$$

where h is the maximum exploitation rate of the herbivore species and c the rate of herbivores needed to reach half of the maximum exploitation. This harvesting function characterizes the behavior of a commercial harvester when its decision to harvest depends on both the revenue and the cost of harvesting. As more species become available, harvesting more at a linear rate may not be profitable due to supply and demand

$$\begin{cases} \dfrac{dR(t)}{dt} = r\left(1 - \dfrac{R(t)}{K}\right)R(t) - \dfrac{aR(t)H(t)}{b + R(t)} \\ \dfrac{dH(t)}{dt} = e\dfrac{aR(t)H(t)}{b + R(t)} - \mu H(t) - \dfrac{hH(t)}{c + H(t)}. \end{cases} \#(1.1)$$

In the first case, we considered a model with continuous harvesting of livestock. This case, although it presents some results, is limited because livestock cannot be harvested continuously. So we're going to move on to a much more realistic situation. Indeed, no species can be harvested continuously, but rather at specific times for known reasons (sale, bequeathing, slaughter, etc.).Many other investigations into impulsive models have been carried out in ecology[4], [8], [11], [12] and [7].With this in mind, we're going to consider a model with impulsive harvesting.

$$\begin{cases} \dfrac{dR(t)}{dt} = r\left(1 - \dfrac{R(t)}{K}\right)R(t) - \dfrac{aR(t)H(t)}{b + R(t)}, \quad t \neq t_k \\ \dfrac{dH(t)}{dt} \quad = e\dfrac{aR(t)H(t)}{b + R(t)} - \mu H(t), & \#(1.2) \\ R(t_k^+) \quad = R(t_k), & t = t_k, t_{k+1} = t_k + \tau, k = 1,2, \dots, \\ H(t_k^+) \quad = \left(1 - \omega\left(H(t_k)\right)\right)H(t_k), \end{cases}$$

where, $\omega(H) = \frac{hH}{c+H}$.

2.Main Results

2.1 Results Of System (1.1)

We start by showing that solutions of system (1.1) that start in R_+^2 will remain there and are uniformly bounded. The following Lemma 2.1 is valide. Its proof can be done as in Tankam et al. (2015)[15].

Lemma 2.1

If $\boldsymbol{r} \leq \boldsymbol{0}$, then all the tmjectories of the System (1.1) go to the zero solution.

If $\boldsymbol{r} > 0$,

(i) The non negative orthant $\mathbb{R}_+^2$ is a positively invariant region for the System (1.1).

(ii) the set $\boldsymbol{\Omega}$ defined by:

$$\Omega = \left\{(R, H) \in \mathbb{R}_+^2, R \leq K\right\}$$

is a positively invariant and absorbing set for the System (1.1).

From Lemma 1 we can deduce that system (1.1) is biologically well-posed. Furthermore, we also deduce that system (1.1) is a dissipative system (Hale 1988)[14].

Proposition 2.2 (Equilibria of the Model)

1. The trivial equilibria point $E_0 = (0,0)$ and $E_1 = (K, 0)$ which always exists.
2. If $r > 0$, the system (1.1) admits at least one coeristence equilibrium $E^* = (R^*, H^*)$ where $H^* = \frac{r(b+R^*)(K-R^*)}{aK}$ with $K > R^*$ and R^* is a positive solution of the following equation:

$$a_1R^{*3} + a_2R^{*2} + a_3R^* + a_4 = 0. \#(2.1)$$

The values $a_1, a_2,$ a_2 and a_4 are defined in the proof.

Proof. Equilibria in System (1.1) are obtained by solving the following system:

$$\begin{cases} r\left(1 - \dfrac{R(t)}{K}\right)R(t) - \dfrac{aR(t)H(t)}{b + R(t)} = 0 \\ \epsilon\dfrac{aR(t)H(t)}{b + R(t)} - \mu H(t) - \dfrac{hH(t)}{c + H(t)} = 0. \end{cases}$$

1. The trivial equilibria of system (2.1) are given by: $E_0 = (0,0)$ and $E_1 = (K, 0)$.
2. When $R \neq 0$ and $H \neq 0$ then from the first equation of system (1.1) we have,

$$H^* = \frac{r(b + R^*)(K - R^*)}{aK}$$

Replacing H^* by its expression in the second equation of (1.1) yields

$$\mathbf{a_1R^3 + a_2R^2 + a_3R^* + a_4 = 0\#(2.2)}$$

where, $\mathbf{a_1 = rea\left(\frac{\mu}{ea} - 1\right); a_3 = (2\mu brK + ahK + ea\mu c)(\mathcal{S}_1 - 1); a_2 = r(eab + \mu K)(\mathcal{S}_2 - 1)}$ and $\mathbf{a_4 = -(abc\mu K + aKhb + rb^2\mu K)}$; with $\mathcal{S}_1 = \frac{\mathbf{(ea^2cK+rbeaK+r\mu b^2)}}{\mathbf{(2\mu brK+ahK+ea\mu c)}}$ and $\mathcal{S}_2 = \frac{\mathbf{(2\mu b+eaK)}}{\mathbf{eab+\mu K}}$. By Descartes' rule of signs, if $\mathcal{S}_1 > 1$ and $\mathcal{S}_2 > 1$ then equation (2.2) admits at least one positive solution $\mathbf{R}^*$.

This result addresses the problem of characterizing the stability of the various equilibria in model (1.1).

Theorem 2.3 (stability of equilibria)

1.$\mathbf{E_0}$ is locally asymptotically stable if $\mathbf{r < 0}$.

2. $\mathbf{E_1}$ is locally asymptotically stable if $\mathbf{r > 0}$ and $\mathbf{S_0 < 1}$where, $\mathbf{S_0} = \frac{\mathbf{ceaK}}{\mathbf{(b+K)(c\mu+h)}}$.

3. The coexistence equilibrium $\mathbf{E^+}$is locally asymptotically stable if the following Routh-Hurwitz conditions are satisfied $\mathbf{a_{11} + a_{22} < 0}$ and $\mathbf{a_{11}a_{22} > a_{12}a_{21}}$ The values $\mathbf{a_{ij}}$ are given in the proof.

Proof.

1. It is easy to verify that the eigenvalues of the Jacobian matrix at $\mathbf{E_0 = (0, 0)}$ are $\mathbf{r}$ and $-\left(\boldsymbol{\mu} + \frac{\mathbf{h}}{\mathbf{c}}\right)$. Therefore

- if $\mathbf{r < 0}$, then $\mathbf{E_0}$ is locally asymptotically stable.
- if $\mathbf{r > 0}$, then $\mathbf{E_0}$ is a saddle-point and therefore unstable.

2. The eigenvalues of the Jacobianmatrix at $\mathbf{E_1 = (K, 0)}$ are $\mathbf{-r}$ and $\frac{\mathbf{c\mu+h}}{\mathbf{c}}\mathbf{(S_0 - 1)}$, where $\mathbf{S_0} = \frac{\mathbf{ceaK}}{\mathbf{(b+K)(c\mu+h)}}$. Hence, equilibrium $\mathbf{E_1}$ is locally asymptotically stable if $\mathbf{S_0 < 1}$.
3. The characteristic polynomial associated with the Jacobian matrix at the equilibrium point $\mathbf{E}^*$ is given by:

$$\mathbf{P(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} + a_{12}a_{21}\#(2.3)}$$

$$\mathbf{a_{11} = r\left(1 - \frac{2R_i^*}{K}\right) - \frac{abH_i^*}{(b + R_i^*)^2}, a_{12} = \frac{aR_i^*}{b + R_i^*},}$$

$$\mathbf{a_{21} = \frac{\epsilon abH_i^*}{(b + R_i^*)^2} a_{22} = \frac{\epsilon aR_i^*}{b + R_i^*} - \mu - \frac{hc}{(c + H_i^*)^2}.}$$

Therefore $\mathbf{E}^*$ is locally asymptotically stable if $\mathbf{a_{11}} + \boldsymbol{a_{22}} < \mathbf{0}$ and $\boldsymbol{a_{11}a_{22} + a_{12}a_{21}} > \mathbf{0}$.

Pastoral Interpretation OfThe Results

1. Stability of equilibrium $E_0 = (0,0)$ (absence of the resource and livestock in the area under consideration) corresponds to the situation that occurs when pastoralists migrate with their herds (transhumance phenomenon). This situation is due to low rainfall in the Sahelian zone, which leads to drought.

2. In the absence of herbivores, depending on certain climatic changes (r>0 and S_0 <1), the forage resource tends towards the capacity limit of the environment.

As regards the stability of the coexistence equilibrium, we have a peak for the resource trajectory, followed by a peak for the herd trajectory.

Theorem 2.4 (Hopf bifurcation) The System (1.1) has a Hopf bifurcation around coexistence equilibrium $\boldsymbol{E}^* = (\boldsymbol{R}^*, \boldsymbol{H}^*)$ when e passes through $\boldsymbol{e_c}$, where $\boldsymbol{e_c} = \frac{\boldsymbol{b+R^*}}{\boldsymbol{aR^*}}(\boldsymbol{\delta^* - r})$, with

$\boldsymbol{\delta^* = \frac{2R_i^*}{K} + \frac{arbH_i^*}{(b+R_i^*)^2} + \mu + \frac{hc}{(c+H_i^*)^2}}$.

The pastoral interpretation of the Hopf bifurcation is that herbivores will coexist with the resource,

2.2 Results of System (1.2)

2.2.1 Existence, positivity and boundedness of solutions

The right-hand side of system (1.2) is locally lipschitz continuous on $\mathbb{R}^2$. Thus, using a classic existence theorem (Theorem 1.1, p. 3 in Bainov and Simeonov (1995)), there exist $\boldsymbol{\alpha > 0}$ and a unique solution defined from $(\mathbf{0}, \boldsymbol{\alpha}) \to \mathbb{R}^2$ for system (1.2).

System (1.2) is designed to model resource-herbivore interactions. It is therefore important that its solutions remain positive and bounded, so that our system is biologically well-posed.
Lemma 2.5 Thepositive orthant R_{+}^{2} is a positively invariant region for the system (1.2).

2.2.2 Trivial AndSemitrivial Solutions

We determine the resource-free periodic solution and give its stability condition. In the absence of the resource, system (1.2) becomes:

$$\begin{cases} \frac{dH(t)}{dt} = \mu H(t), & t \neq t_k \\ H(t_k^{+}) = \left(1 - \omega\big(H(t_k)\big)\right) H(t_k), & t = t_k, t_{k+1} = t_k + \tau, k = 1, 2, 3 \ldots \end{cases} \quad (2.4)$$

with, $0 \leq H(0) = H_0$.
Lemma 2.6
System (2.4) has a positive herbivores land periodic solution $E_H = (0, H^*(t))$.

$$H^*(t) = H^*(\theta^{+}) e^{-\mu(t - n\tau^{*})}, \forall t \in [n\tau, (n+1)\tau],$$

where,

$$H^*(0^{+}) = e^{\mu\tau} \frac{c(1 - e^{\mu T})}{h - 1 + e^{\mu\tau}}.$$

Proof. Let $t \in |n\tau, (n+1)\tau|$, by integrating the first equation of system (2.4) between $n\tau^{+}$and t we have:

$$\int_{n\tau^{+}}^{t} \frac{dH}{H} = -\mu \int_{n\tau^{+}}^{t} d\mathrm{t}$$

$$H(t) = H(n\tau^{+}) e^{-\mu(t - n\tau)}.$$

Using the second equation of system (2.4), we obtain

$$H(t) = \left(1 - \frac{hH(n\tau)}{c + H(n\tau)}\right) H(n\tau) e^{-\mu(t - n\tau)}, \qquad \forall t \in [n\tau, (n+1)\tau].$$

$H(n\tau)$ is in $](n-1)\tau, n\tau]$, so to obtain it, we integrate the first equation of system (2.4) between $(n-1)\tau^{+}$and t, i.e.

$$\int_{(n-1)\tau+}^{t} \frac{dH}{H} = -\mu \int_{(n-1)\tau+}^{t} dt,$$

$$H(t) = H((n-1)\tau^{+}) e^{-\mu(t - (n-1)\tau)}.$$

Hence,

$$H(t) = \left(1 - \frac{hH\big((n-1)\tau\big)}{c + H\big((n-1)\tau\big)}\right) H\big((n-1)\tau\big) e^{-\mu(t - (n-1)\tau)}, t \in \Big] (n-1)\tau, n\tau \Big].$$

Therefore

$$H(n\tau) = \left(1 - \frac{hH((n-1)\tau)}{c + H((n-1)\tau)}\right) H((n-1)\tau) e^{-\mu\tau}.$$

Let $U_n = H(n\tau)$, then $U_n = F(U_{n-1})$ where $F(x) = x\left(1 - \frac{hx}{c+x}\right) e^{-\mu\tau}$. Moreover we have $F(H(n\tau)) = H(n\tau) \Rightarrow H(n\tau)\left(1 - \frac{hH(n\tau)}{c+H(n\tau)}\right) e^{-\mu\tau} = H(n\tau)$. Since $H(n\tau) \neq 0$, then $H(n\tau) = \frac{c(1-e^{\mu\tau})}{h+e^{\mu\tau}-1}$.

Thus,

$$H^*(t) = e^{\mu\tau} \frac{c(1 - e^{\mu\tau})}{h - 1 + e^{\mu\tau}} e^{-\mu(t - n\tau +)}, \forall t \in]n\tau, (n+1)\tau]. \qquad (2.5)$$

We have

$$H(0^{+}) = H(n\tau^{+}),$$
$$H^*(0^{+}) = H^{'}(n\tau^{+}),$$
$$H^*(0^{+}) = e^{\mu\tau} \frac{c(1 - e^{\mu\tau})}{h - 1 + \epsilon^{\mu\tau}}.$$

Let's show that the solution $(0, H^*)$ is periodical with period τ. Using (2.5) we have:

$$H^*((n+1)\tau) = e^{\mu\tau} \frac{c - e^{\mu\tau}}{h - 1 + e^{\mu\tau}} e^{-\mu\tau},$$

$$H^*((n+1)\tau) = \frac{c - e^{\mu\tau}}{h - 1 + e^{\mu\tau}}.$$

On the other hand, we also have:

$$H^*(nT) = \frac{c - e^{\mu T}}{h - 1 + e^{\mu T}}.$$

Therefore

$$H^*\big((n+1)\tau\big) = H^*(n\tau). \qquad (2.6)$$

So the resource-free solution $(0, H^*(t))$ with initial condition $(0, H^*(0^+)$is periodic with period τ.

Now well show the local stability of this positive, periodic solution for herbivores. To do this, we'll use the small perturbation technique and Floquet theory (Hale 1980; D'Onofrio 2002; Chen et al. 2009), i.e. we'll find conditions under which all Floquet multipliers of the positive periodic solution have their absolute value less than or equal to unity (Hale 1980; D'Onofrio 2002; Chen et al. 2009).

Theorem 2.7: The solution $(0, H^*)$ is locally asymptotically stable if and only if:

$e > \frac{rb}{H^*}$.

Proof. We linearize system (2.4) in the neighbourhood of $(0, H^*)$ by applying the following change of variables:

$$\begin{cases} x(t) = R(t) \\ y(t) = H(t) - H^*(t) \end{cases}$$

where, $x(t)$ and $y(t)$ are small perturbations in the vicinity of the periodic solution and satisfy

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

where φ is a fundamental matrix and satisfie:

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} r - \dfrac{eH^*}{b} & 0 \\ \dfrac{eaH^*}{b} & -\mu \end{pmatrix} \Phi(t)$$

and $\Phi(0) = Id_2$. Furthermore, there setting of impulsive condition of system (2.4) becomes;

$$\begin{pmatrix} x(n\tau^+) \\ y(n\tau^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1-h \end{pmatrix} \begin{pmatrix} x(n\tau) \\ y(n\tau) \end{pmatrix} = (B_k + I) \begin{pmatrix} x(n\tau) \\ y(n\tau) \end{pmatrix}.$$

A monodromy matrix M of system (2.4), is:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1-h. \end{pmatrix} \Phi(t).$$

Thus, the eigenvalues of M are: $\mu_1 = e^{+\left(r - \frac{eH^*}{b}\right)}$; and $\mu_2 = (1-h)e^{-\mu\tau} < 1$. Therefore, (Floquet's Theory)[3] the solution $(0, H^*(t))$ is locally asymptotically stable if and only if :

$$e > \frac{br}{H^*}.$$

2.2.3 Existence of a positive and periodic resource-herbivores solution

Now we show the existence of at least one non-trivial positive periodic solution of system (1.2). We will use the approach developed by Gaines and Mawhin (1977)[6].

Theorem 2.8 System (1.2) admits a positive τ-periodic solution (R^*, H^*) if and only if :

$$t_k > max\left\{\frac{\ln\big(1 - \omega(H_1(\tau))\big)}{r}, K\right\} \Leftrightarrow \tau > max\left\{\frac{\ln\big(1 - \omega(H_1(\tau))\big)}{nr}, \frac{K}{n}\right\}.$$

Proof. To show the existence of the τ-periodic solution we will prove that all the assumptions of the contimuation theorem are satisfied by proceeding as follows:

Taking new variables $R(t) = e^{x(t)}$ and $H(t) = e^{y(t)}$, then system (2.4) becomes:

$$\begin{cases} \dfrac{dx}{dt} = r\left(1 - \dfrac{e^{x(t)}}{K}\right) - \dfrac{ae^{y(t)}}{b + e^{x(t)}}, \quad t \neq t_k \\ \dfrac{dy}{dt} = e_k \dfrac{ae^{x(t)}}{b + e^{x(t)}} - \mu, \\ x(t_k^+) = x(t_k), \qquad t = t_k, t_{k+1} = t_k + \tau, k = 1, 2, \dots, \\ y(t_k^+) = \ln\Big(1 - \omega\big(y(t_k)\big)\Big) + y(t_k), \end{cases} \#(2.7)$$

with $0 \le x(0) = ln(R_0)$, $0 \le y(0) = \ln(H_0)$.

Let

$X = C^1\big([0, \tau], \mathbb{R}^2\big)$ and $Y = C^1\big([0, \tau], \mathbb{R}^2\big) \times C^1\big([0, \tau], \mathbb{R}^2\big)$ (2.8) and for

$$\boldsymbol{u = (x, y) + \|u\| = \max_{t\in[0,T]}|x(t)| + \max_{t\in[0,r]}|y(t)|.}$$

Then $\boldsymbol{X, Y}$ are Banach spaces when they are endowed with the above norm $\|.\|$. Let,

$$\boldsymbol{L: dom(L) \to Y}$$

$$\boldsymbol{\begin{pmatrix} x \\ y \end{pmatrix} \to \left(\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}, \begin{pmatrix} \Delta x(t_k) \\ \Delta y(t_k) \end{pmatrix}\right),}$$

$$\boldsymbol{N\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \left(N_1\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, N_2\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\right),}$$

with:

$$\boldsymbol{N_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\left(1-\frac{e^{x(t)}}{K}\right)-\frac{ae^{y(t)}}{b+e^{x(t)}} \\ -\mu+e\frac{ae^{x(t)}}{b+e^{x(t)}} \end{pmatrix}} \text{ and } \boldsymbol{N_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ ln\left(1-\frac{h}{1+ce^{-y(\tau)}}\right) \end{pmatrix}.}$$

A direct computation leads to

$$\boldsymbol{KerL = \left\{\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \mathbb{R}^2\right\}}$$

and

$$\boldsymbol{ImL = \left\{\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right) \in Y: \begin{pmatrix} \int_0^{t_t} x(t)dt + a = 0 \\ \int_0^{t_*} y(t)dt + b = 0 \end{pmatrix}\right\}.}$$

Since $\boldsymbol{ImL}$ is closed in Z, L is a Fredholm mapping and

$$\begin{aligned} \boldsymbol{index(L)} &= \boldsymbol{dim}(\text{ Ker } \boldsymbol{L}) - \boldsymbol{dim}(\text{ CoKer } \boldsymbol{L}) \\ &= \boldsymbol{dim}(\text{ Ker } \boldsymbol{L}) - [\boldsymbol{dim(Y) - dim(ImL)}]. \\ &= \boldsymbol{2 - (4 - 2)} \\ &= \boldsymbol{0.} \end{aligned}$$

Then, $\boldsymbol{L}$ is a Fredholm operator of index zero. Thus following (Gaines and Mawhin 1977, P.12)[6], there exist two continuous projectors P and Q such that the sequel $\boldsymbol{X \overset{P}{\to} domL \overset{L}{\to} Y \overset{Q}{\to} Y}$ is exact i.e. $\boldsymbol{ImP = KerL}$ and $\boldsymbol{KerQ = ImL = Im(I - Q)}$. It suffices to choose

$$\boldsymbol{P\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix}}$$

and

$$\boldsymbol{Q\left(\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}\right) = \left(\frac{1}{\tau}\begin{pmatrix} \int_0^{\tau} x(t)dt+a=0 \\ \int_0^{T} y(t)dt+b=0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right).}$$

We check that

$$\boldsymbol{LP\begin{pmatrix} x \\ y \end{pmatrix} = L\begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = 0_Y}$$

$$\begin{aligned} \boldsymbol{QL\begin{pmatrix} x \\ y \end{pmatrix}} &= \boldsymbol{Q\left(\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \cdot \begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix}\right)} \\ &= \boldsymbol{\left(\frac{1}{\tau}\begin{pmatrix} \int_0^{\tau} x(t)dt + x(\tau) = 0 \\ \int_0^{T} y(t)dt + y(\tau) = 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)} \\ &= \boldsymbol{\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right).} \end{aligned}$$

Furthermore the generalized inverse $\boldsymbol{K_p: ImL \to KerP \cap DomL}$ of the map $\boldsymbol{L: KerP \cap DomL \to ImL}$ is given by

$$\boldsymbol{K_p\left(\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{pmatrix} \int_0^t x(t)dt + a \\ \int_0^t y(t)dt + b \end{pmatrix}.}$$

Indeed, let $\boldsymbol{u = (u_1, u_2)^T \in KerP \cap DomL. (g, r) = \big((g_1, g_2), (r_1, r_2)\big) \in ImL}$.

$$\begin{aligned}
K_pL(u(t))&= K_p(\dot{u},\Delta u),\\
&=\int_0^t u ds+\Delta u,\\
&=u(t)-u(0)+u(0)-u(t_k),\\
&=u(t)-P(u),\\
&=u(t)\text{ since }u\in KerP
\end{aligned}$$

and

$$\begin{aligned}
LK_p(g(t),r)&= L\int_0^t g(s)ds+r_+\\
&=\left(g(t),-\int_0^t g(s)ds\right),\\
&=(g(t),r),\text{ since }(g,r)\in ImL.
\end{aligned}$$

Thus

$$QN\begin{pmatrix}x\\y\end{pmatrix}=\left(\begin{pmatrix}A_1\\A_2\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)$$

with

$$A_1=\frac{1}{\tau}\left(\int_0^\tau\left(r\left(1-\frac{e^{x(t)}}{K}\right)-\frac{ae^{y(t)}}{b+e^{x(t)}}\right)dt+0\right),$$

$$A_2=\frac{1}{\tau}\left(\int_0^\tau\left(-\mu+\frac{ae^{x(t)}}{b+e^{x(t)}}\right)dt+\ln\left(1-\frac{h}{1+ce^{-y(t)}}\right)\right).$$

Therefore

$$\begin{aligned}
K_p(I-Q)N\begin{pmatrix}x(t)\\y(t)\end{pmatrix}&= K_pN\begin{pmatrix}x(t)\\y(t)\end{pmatrix}-K_pQN\begin{pmatrix}x(t)\\y(t)\end{pmatrix},\\
&=\begin{pmatrix}B_1-C_1\\B_2-C_2\end{pmatrix}
\end{aligned}$$

with:

$$B_1=\int_0^t\left(\left(r\left(1-\frac{e^{x(s)}}{K}\right)-\frac{ae^{y(s)}}{b+e^{x(s)}}\right)\right)ds,$$

$$B_2=\int_0^t\left(-\mu+\frac{ae^{x(s)}}{b+e^{x(s)}}\right)ds+\ln\left(1-\frac{h}{1+ce^{-y(t)}}\right),$$

$$C_1=\int_0^t A_1ds=tA_1\text{ and }C_2=\int_0^t A_2ds=tA_2.$$

QN and $KP(IQ)N$ are continuous then for any open bounded set $\Omega\subset X, QN(\bar{\Omega})$ is bounded. Furthermore, let $t_1,t_2\in[0,\tau], u(t)=(x,y)(t)$,

$$f(t,u(t))=\begin{pmatrix}r\left(1-\frac{e^{x(t)}}{K}\right)-\frac{ae^{y(t)}}{b+e^{x(t)}}\\-\mu+\frac{ae^{x(t)}}{b+e^{x(t)}}\end{pmatrix}$$

and

$$a_1=\begin{pmatrix}0\\\ln\left(1-\frac{h}{1+ce^{-y(t)}}\right)\end{pmatrix}.$$

Let $H=K_P(I-Q)N$ we have:

$$|Hu(t_2)-Hu(t_1)|=\left|\begin{pmatrix}B_1\\B_2\end{pmatrix}(t_2)-\begin{pmatrix}C_1\\C_2\end{pmatrix}(t_2)-\begin{pmatrix}B_1\\B_2\end{pmatrix}(t_1)-\begin{pmatrix}C_1\\C_2\end{pmatrix}(t_1)\right|,$$

with:

$$\begin{pmatrix}B_1\\B_2\end{pmatrix}(t_2)=\int_0^{t_2}f(s,u(s))ds+a_1\text{ and }\begin{pmatrix}C_1\\C_2\end{pmatrix}(t_2)=\frac{t_2}{\tau}\left(\int_0^{t_2}f(s,u(s))ds+a_1\right).$$

Let

$$\bar{g} = \frac{t_2}{\tau}\left(\int_0^\tau f(s,u(s))ds + a_1\right) \text{ and } \bar{h} = \frac{t_1}{\tau}\left(\int_0^\tau f(s,u(s))ds + a_1\right).$$

$$\left|H(u(t_2)) - H(u(t_1))\right| = \left|\int_0^{t_2} f(s,u(s))ds - \int_0^{t_1} f(s,u(s))ds - \bar{g} + \bar{h}\right|,$$

$$= \left|\int_0^{t_2} f(s,u(s))ds - \frac{t_2 - t_1}{\tau}\left(\int_0^\tau f(s,u(s))ds + a_1\right)\right|.$$

Therefore, we have:

$$\left|H(u(t_2)) - H(u(t_1))\right| \leq |t_2 - t_1| \max_{t\in[0,\tau]} |f(t,u(t))| + \frac{t_2 - t_1}{\tau}\left(\tau \max_{t\in[0,\tau]} |f(t,u(t) + a_1)|\right)$$

thus,

$$\left|H(u(t_2)) - H(u(t_1))\right| \leq 2|t_2 - t_1| \max_{t\in[0,\tau]} |f(t,u(t))| + \frac{a_1}{\tau}. \qquad (2.9)$$

On the other hand,

$$|H(t)| = \left|\int_0^t f(s,u(s))ds + a_1 - \frac{t}{\tau}\left(\int_0^t f(s,u(s))ds + a_1\right)\right|$$

and since $t \in [0,\tau]$, we have:

$|H(t)| \leq \left|\int_0^\tau f(s,u(s))ds + a_1 + \left(\int_0^\tau f(s,u(s))ds + a_1\right)\right|.$

Thus:

$$\begin{aligned} |H(t)| &\leq \tau \max_{s\in[0,\tau]} |f(s,u(s))| + |a_1| + \tau \max_{s\in[0,\tau]} |f(s,u(s))| + |a_1|\Big) \\ &\leq 2\left(\tau \max_{s\in[0,\tau]} |f(s,u(s))| + |a_1|\right) \\ |H(t)| &\leq 2\left(\tau \max_{s\in[0,\tau]} \left|f(s,u(s))\right| + |a_1|\right). \qquad (\mathbf{2.10}) \end{aligned}$$

Then using relations (2.9), (2.10) and the Arzela-Ascoli's theorem (Sonntag 1997, Theo rem 3.1, p.314) [5] we deduce that $KP(I-Q)N(\bar{\Omega})$ is compact. Thus, N is a L-compact mapping on $\bar{\Omega}$. The isomorphism J of ImQ on to KerL may be defined by:

$$\begin{aligned} J = ImQ &\longrightarrow kerL \\ \left(\begin{pmatrix}u\\v\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right) &\mapsto \begin{pmatrix}u\\v\end{pmatrix}. \end{aligned}$$

Now we reach the position to search for an appropriate open, bounded subset Ω for the application of the continuation theorem, i.e. we search M_0 such that every τ-periodic solution of system (2.4) satisfies: $|x(t)| + |y(t)| \leq M_0$, with $0 \leq t \leq \tau$.

Corresponding to the operator equation $Lx = \lambda Nx, \lambda \in (0,1)$, we have:

$$\begin{cases} \frac{dx(t)}{dt} &= \lambda\left(r\left(1 - \frac{e^{x(t)}}{K}\right) - \frac{ae^{y(t)}}{b+e^{x(t)}}\right),\ t \neq t_k \\ \frac{dy(t)}{dt} &= \lambda\left(\frac{eae^{x(t)}}{b+e^{x(t)}} - \mu\right). \\ x(t_k^+) - x(t_k) &= 0, \\ y(t_k^+) - y(t_k) &= \lambda\left(ln\left(1 - \omega(y(t_k))\right)\right). \end{cases} \quad t = t_k, t_{k+1} = t_k + \tau, k = 1,2,., \qquad (2.11)$$

Suppose that $(x(t), y(t)) \in X$ is an arbitrary solution of system (2.7) for acertain $\lambda \in (0,1)$. Integrating on both sides of (2.7) over the interval $[0,\tau]$, we obtain:

$$\begin{cases} x(t) = x(0) + \lambda\left(\int_0^t \left(r\left(1 - \frac{e^{x(s)}}{K}\right) - \frac{ae^{y(s)}}{b+e^{x(s)}}\right)ds + 0\right) \\ y(t) = y(0) + \lambda\left(\int_0^t \left(\frac{eae^{x(s)}}{b+e^{x(s)}} - \mu\right)ds + ln\left(1 - \omega(y(t_k))\right)\right). \end{cases}$$

This implies,

$$\begin{cases} \int_0^\tau \left(r\frac{e^{x(t)}}{K} + \frac{ae^{y(t)}}{b+e^{x(t)}}\right)dt = r\tau \\ \int_0^\tau \frac{eae^{x(t)}}{b+e^{x(t)}} \mathrm{dt} = \mu\tau - ln\left(1 - \omega(y(\tau))\right). \end{cases} \qquad (2.12)$$

Since X is a Banach space and $(x(t), y(t)) \in X$, there exist $\bar{\varepsilon}$, and ς such that $x(\bar{\varepsilon}) = \max_{t\in[0,\tau]} x(t)$, $x(\underline{\varepsilon}) = \min_{t\in[0,\tau]} x(t), y(\bar{\varepsilon}) = \max_{t\in[0,\tau]} y(t)$ and $y(\underline{\varepsilon}) = \min_{t\in[0,\tau]} y(t)$.

On the other hand:

$\int_0^\tau |x'(t)dt| \leq r\tau + \int_0^\tau \left(r\frac{e^{x(t)}}{K} + \frac{ae^{x(t)}}{b+e^{x(t)}}\right) dt$.

Thus, from (2.9), we have

$\int_0^\tau |x'(t)|dt \leq 2r\tau = M_{x'}$. (2.13)

Furthermore,

$$\int_0^\tau |y'(t)|dt \leq \tau\mu + \int_0^\tau \left|\frac{eae^{x(t)}}{b+e^{x(t)}}\right| dt$$

we have

$\int_0^\tau |y'(t)|dt \leq 2\tau\mu + ln(1 - \omega(y(\bar{\varepsilon}))) = M_{y'}$. (2.14)

We can also write:

$\int_0^\tau r\frac{e^{x(t)}}{r}\frac{r\tau}{\frac{r\tau}{K}e^{x(\varepsilon)}} dt$,

$$x\left(\underline{\varepsilon}\right) \leq ln\, K, \qquad (2.15)$$

$$\int_0^\tau \frac{ae^{y(t)}}{b+e^{x(t)}} dt \leq rt_{k;}$$

$$\Rightarrow e^{x(y(\varepsilon)}\leq \frac{b+e^{\underline{\varepsilon}}}{a} r,$$

$$\Rightarrow y(\underline{\varepsilon})\leq ln\left[\frac{b+e^{x(\xi)}}{a} r\right],$$

$$\Rightarrow y(\underline{\varepsilon})\leq ln\left[\frac{r\left(b+e^{x(\xi)}\right)}{a}\right].$$

Thus,

$y(\underline{\varepsilon}) \leq ln\left[\frac{r(b+K)}{a}\right] = M_{\underline{\varepsilon}^*}$. (2.16)

In the same way, we have:

$\int_0^\tau eae^{\tau(t)} \geq \mu\tau - ln(1 - \omega(y(\tau)))$.

Furthermore:

$x(\bar{\varepsilon}) \geq ln\left[\frac{\mu\tau - ln(1-\omega(y(\tau)))}{ea\tau}\right]$ if $\tau > \frac{ln(1-\omega(y(\tau)))}{\mu}$, (2.17)

We have also:

$$\int_0^\tau \left(re^{x(t)} + ae^{y(t)}\right) \geq r\tau.$$

We obtain:

$$\left(re^{x(\varepsilon)} + ae^{y(\epsilon)}\right)\geq r\tau,$$

$$e^{y(\varepsilon)}\geq \frac{r\tau}{a}\left(1 - e^{x(\varepsilon)}\right),$$

$$y(\bar{\varepsilon})\geq ln\left(r\tau - re^{\tau(\varepsilon)}\right) - ln\, a.$$

Since $e^y = H < K$ then,

$$y(\bar{\varepsilon}) \geq ln(r\tau - rK) - ln\, a, \text{ for } \tau > K. \qquad (2.18)$$

$\forall t \in [0, \tau]$, on a:

$\int_{\underline{\varepsilon}}^t \frac{dx}{ds} \leq \int_{\underline{\varepsilon}}^t \left|\frac{dx}{ds}\right|$.

Therefore we obtain:

$$x(t) - x(\underline{\underline{\varepsilon}}) \int_0^\varepsilon \left|\frac{dx}{ds}\right| + \int_{\underline{\varepsilon}}^t \left|\frac{dx}{ds}\right| + \int_t^{t_x} \left|\frac{dx}{ds}\right|.$$

Then

$x(t) \leq x(\underline{\varepsilon}) + \int_0^\tau |x'(t)|dt$.

We obtain:

$x(t) \ln K + 2r\tau = x_M.$ (2.19)
We have also:

$$y(t) y(\underline{\varepsilon}) + \int_0^\tau |y'(t)| dt$$

thus:
$y(t) M_{\underline{\underline{E}}} + M_{y'} = y_M.$ (2.20)
Moreover $\forall t, \varepsilon \in [0, \tau]$ with $t < \bar{\varepsilon}$ we obtain:

$$\int_t^t \frac{dy}{ds} \leq \int_t^\varepsilon \left|\frac{dy}{ds}\right|$$

$$\Rightarrow y(\bar{\varepsilon}) - y(t) \leq \int_0^t \left|\frac{dy}{ds}\right| + \int_t^\varepsilon \left|\frac{dy}{ds}\right| + \int_\varepsilon^\tau \left|\frac{dy}{ds}\right|$$

then $y(t) \geq y(\bar{\varepsilon}) - \int_0^\tau |y'(t)|$.

Thus:

$$y(t) \geq m_y - M'_y = y_m. \qquad (2.21)$$

We have also:

$$\int_t^t \frac{dx}{ds} \leq \int_t^t \left|\frac{dy}{ds}\right|,$$

$$\Rightarrow x(\bar{\varepsilon}) - x(t) \leq \int_0^t \left|\frac{dx}{ds}\right| + \int_t^\varepsilon \left|\frac{dx}{ds}\right| + \int_\varepsilon^\tau \left|\frac{dx}{ds}\right|,$$

$$\Rightarrow x(t) \geq x(\bar{\varepsilon}) - \int_0^\tau |x'(t)|.$$

Then:
$x(t) \geq x_m.$ (2.22)
To summarize:

$$\begin{cases} x_m \leq x(t) \leq x_M \\ y_m \leq y(t) \leq y_M. \end{cases}$$

Therefore:

$$\begin{cases} \max_{t \in [0,\tau]} |x(t)| \leq \max\{|x_m|, |x_M|\} = M_x \\ \max_{t \in [0,\tau]} |H_1(t)| \leq \max\{|y_m|, |y_M|\} = M_y. \end{cases} \qquad (2.23)$$

Thus, we have M_y and M_x independent of λ. Then we proceed like Yatat and Tchuinté in [7] and [10] respectively.
Let $M_0 = M_x + M_y$. Then, $\binom{x}{y} \in X$, we have:
$\|(x, y)^T\| < M_0$.
Let $\Omega = \{(x, y)^T \in X : \|(R_1, H_1)\| < M_0\}$, so Ω verifies the first hypothesis of the continuation theorem of Gaines and Mawhin 1977 page 40)[6].
For $\lambda \in [0, 1]$: if $(x, y) \in \partial KerL = \partial\Omega \cap \mathbb{R}^2$, (x, y) is a constant vector in $\mathbb{R}^2$ with, $\|(x, y)\| = M_0$, then we have :

$$QN\begin{pmatrix} x \\ y \end{pmatrix} = (A_1 A_2), (0) \neq ((0), (0)).$$

Consequently, the first part of assumption (2) of the continuation theorem is satisfied. if $(x, y) \in KerL \cap \Omega$, we have :
$JQN(R_1, H_1)^T = \binom{A_1}{A_2}$,
where

$$A_1 = \frac{1}{\tau}\left(\int_0^\tau \left(r\left(1 - \frac{e^{x(t)}}{K}\right) - \frac{ae^{y(t)}}{b + e^{x(t)}}\right) dt\right)$$

$$A_2 = \frac{1}{\tau}\left(\int_0^\tau \left(-\mu + \frac{eae^{x(t)}}{b + e^{x(t)}}\right) dt + \ln\left(1 - \frac{h}{1 + ce^{-y(\tau)}}\right)\right).$$

Then according to Tchuinté in [14] we define:

$$\Psi : domL \times [0, 1] \to X$$

such that:

$$\boldsymbol{\Psi}(x,y,\lambda)=\begin{pmatrix}\frac{1}{\tau}\left(r\left(1-\frac{e^{x(t)}}{K}\right)-\frac{ae^{y(t)}}{b+e^{x(t)}}\right)\\ \frac{1}{\tau}\left(\frac{eae^{x(t)}}{b+e^{x(t)}}-\mu\right)\end{pmatrix}+\lambda\begin{pmatrix}0\\ ln\left(1-\frac{h}{1+ce^{-y(t)}}\right)\end{pmatrix}.$$

Then we know that for $(x,y)\in\Omega\cap KerL$. We have $\Psi((x,y),\lambda)\neq 0$, otherwise there exists a constant vector (x,y) with $\|(x,y)\|=M_0$, implies $\Psi((x,y),\lambda)=0$. i.e.:

$$\begin{cases}\frac{r}{\tau}\frac{e^{x(t)}}{K}+\frac{1}{\tau}\frac{ae^{y(t)}}{b+e^{x(t)}}-\frac{r}{\tau}=0\\ \frac{eae^{x(t)}}{b+e^{x(t)}}+\lambda ln\left(1-\frac{h}{1+ce^{-y(t)}}\right)=0.\end{cases}\qquad(2.24)$$

From all the above, the preceding system (31) is satisfied with the condition $\|(R_1,H_1)\|< M_0$, which contradicts $\|(x,y)\|=M_0$. Thus, according to the theory of the coincidence degree property (Gaines and Mawhin **1977**)[6], we have for: $U=(x,y)^T: deg(JQNu,\Omega\cap KerL,0)=deg(\Psi(U,\lambda),\Omega\cap KerL,0)=deg(\Psi(U,0),\Omega\cap KerL,0)$.

As a result, the first part of the continuation theorem is verified. Consider :

$$H_\lambda((x,y)^T)=\lambda JQN((x,y)^T)+(1-\lambda)D((x,y)^T),\qquad \lambda\in[0,1]$$

with

$$D=\left(\frac{1}{\tau}\left[r\left(1-\frac{e^{x(t)}}{K}\right)-\frac{ae^{y(t)}}{b+e^{x(t)}}\right]\frac{1}{\tau}\left[\left(\frac{eae^{x(t)}}{b+e^{x(t)}}-\mu\right)+ln\left(1-\frac{h}{1+ce^{-y(t)}}\right)\right]\right).$$

We have $0\notin H_\lambda(\partial\Omega\cap KerL)$ for $\lambda\in[0,1]$ the equation $D((x,y)^T)=0$ has a solution $(e^x,e^y)^T\in\mathbb{R}^2$. We calculate the Brower degree $deg(JQN,\partial\Omega\cap KerL,0)$. Using the homotopy [6]. Finally, we'll calculate Brouwer's degrees and conclude.

$$\begin{aligned}deg(JQN,\partial\Omega\cap KerL,0)&=deg(D,\partial\Omega\cap KerLL,0),\\ &=\sum_{p\in v^{-}(0)}Sign(J_D(p)),\\ &=Sign\left\{det\begin{pmatrix}-\frac{re^{R_1(t)}}{K}+\frac{ae^{H_1(t)}}{(b+e^{R_1(t)})^2} & -\frac{ae^{H_1(t)}}{b+e^{R_1(t)}},\\ \frac{eae^{H_1(t)}}{b+e^{R_1(t)}} & 0\end{pmatrix}\right\},\\ &=Sign\left(\frac{eae^{H_1(t)}}{b+e^{R_1(t)}}\times\frac{ae^{H_1(t)}}{b+e^{R_1(t)}}\right),\\ &=1,\\ &\neq 0.\end{aligned}$$

Thus,

$$deg(JQN,\partial\Omega\cap KerL,0)=1\neq 0.\qquad(2.25)$$

Subsequently, as Ω satisfies all the assumptions of the continuation theorem (Gaines and Mawhin 1977)[6], so $Lx=Nx$ has at least one solution in $dom(L)\cap\bar{\Omega}$. Consequently the system (2.4) admits at least one solution (x^*,y^*) in $dom(L)\cap\Omega$. Hence (R^*,H^*) is a periodic solution of the system (1.2). This completes the proof.

Now, we study the stability behavior of periodic resource-herbivore solution of system (1.2).

Theorem 2.9:

The periodic resource-herbivore (R^*,H^*) solution of system (1.2) is globally stable.

Proof. Since $R^*(t)$ and $H_*(t)$ are continuous on the compact $[0,\tau]$, then there exist m_R, m_H, M_R and M_H positive such that:

$$\begin{cases}m_R\ \le R^*(t)\le M_R,\\ m_H\le H^*(t)\le M_H.\end{cases}\qquad(2.26)$$

Consider the following Lyapunov function:

$$V(t,R(t),H(t))=\left|ln\frac{R(t)}{R^*(t)}\right|+\left|ln\frac{H(t)}{H^*(t)}\right|.\qquad(2.27)$$

Let $t\in[0,\tau]$, and without loss of generality, assume $R^*(t)<R(t)$. Since the function $s\to ln\, s$ continuous and derivable on $[R^+(t),R(t)]$, then by the mean theorem there exists $c\in[R^+(t),R(t)]$ such that:

$$|ln\, R(t)-ln\, R^*(t)|=\frac{1}{c}|(R(t)-R^*(t))|.$$

Then using (2.27) we obtain

$\frac{1}{M_R}|R(t)-R^*(t)| \le |\ln R(t) - \ln R^*(t)| \le \frac{1}{m_R}|R(t)-R^*(t)|$.

Similarly, we also have:

$$\frac{1}{M_H}|H(t)-H^*(t)| \le |\ln H(t) - \ln H^*(t)| \le \frac{1}{m_H}|H(t)-H^*(t)|$$

i.e.

$$\begin{cases} \frac{1}{M_R}|R(t)-R^*(t)| \le |\ln R(t) - \ln R^*(t)| \le \frac{1}{m_R}|\ln R(t) - \ln R^*(t)| \\ \frac{1}{M_H}|H(t)-H^*(t)| \le |\ln H(t) - \ln H^*(t)| \le \frac{1}{m_H}|H(t)-H^*(t)|. \end{cases} \quad (2.28)$$

From (2.28) we have:

$$\begin{aligned} V(t_0^+, R(t_0^+), H(t_0^+)) &= |\ln R(t_0^+) - \ln R^*(t_0^+)| + |\ln H(t_0^+) - \ln H^*(t_0^+)| \\ &\le \frac{1}{m_R}|R(t_0) - R^*(t_0)| + \frac{1}{m_H}|H(t_0) - H^*(t_0)| \\ &\le \frac{1}{m_R}\|R(t_0) - R^*(t_0)\| + \frac{1}{m_H}\|H(t) - H^*(t)\|. \end{aligned}$$

For $t > 0$ and $t = t_k, k = 1, 2$ ..., we have

$$\begin{aligned} V(t_k^+, R(t_k^+), H(t_k^+)) &= \left|\ln \frac{R(t_k^+)}{R^*(t_k^+)}\right| + \left|\ln \frac{H(t_k^+)}{H^*(t_k^+)}\right| \\ &= \left|\ln \frac{R(t_k)}{R^*(t_k^+)}\right| + \left|\ln \frac{\left(1 - \omega(H(t_k))\right) H(t_k)}{H^*(t_k^+)}\right| \\ &\le \left|\ln \frac{R(t_k)}{R^*(t_k^+)}\right| + \left|\ln \frac{H(t_k)}{H^*(t_k^+)}\right| \\ V(t_0^+, R(t_0^+), H(t_0^+)) &\le V(t_k, R(t_k), H(t_k)). \end{aligned}$$

Thus

$V(t_k^+, R(t_k^+), H(t_k^+)) \le V(t_k, R(t_k), H(t_k))$. (2.29)

Let $D^+V(t, R(t), H(t))$ be the dini derivative of the function $V(t, R(t), H(t))$ with respect to system (1.2).

For $t \ge 0, t \ne t_k, k = 1, 2, \ldots$

$$D^+V(t, R(t), H(t)) = \left(\frac{\dot{R}(t)}{R(t)} - \frac{\dot{R}^*(t)}{R(t)}\right) sign(R(t) - R^*(t)) + \left(\frac{\dot{H}(t)}{H(t)} - \frac{\dot{H}^*(t)}{H(t)}\right) sign(H(t) - H^*(t)).$$

System (1.2) yields

$\frac{\dot{R}(t)}{R(t)} = r\left(1 - \frac{R}{K}\right) - \frac{aH}{b+R}$ and $\frac{\dot{H}(t)}{H(t)} = e\frac{aR}{b+R} - \mu$.

Then

$$\begin{cases} \left(\frac{\dot{R}(t)}{R(t)} - \frac{\dot{R}^*(t)}{R(t)}\right) = r\left(1 - \frac{R(t)}{K}\right) - r\left(1 - \frac{R^*(t)}{K}\right) - \left(\frac{aH(t)}{b+R(t)} - \frac{aH^*(t)}{b+R(t)}\right) \\ \left(\frac{\dot{H}(t)}{H(t)} - \frac{\dot{H}^*(t)}{H(t)}\right) = \frac{e+a}{b+R(t)}(R(t) - R^*(t)), \end{cases}$$

i.e.

$$\begin{cases} \left(\frac{\dot{R}(t)}{R(t)} - \frac{\dot{R}^*(t)}{R(t)}\right) = -\frac{r}{K}(R(t) - R^*(t)) - \frac{a}{b+R(t)}(H(t) - H^*(t)). \\ \left(\frac{\dot{H}(t)}{H(t)} - \frac{\dot{H}^*(t)}{H(t)}\right) = \frac{e+a}{b+R(t)}(R(t) - R^*(t)). \end{cases}$$

Therefore

$$\begin{aligned} D^+V(t,R(t),H(t)) &= -\frac{r}{K}(R(t)-R^*(t))sign(R(t)-R^*(t)) - a\left(\frac{H(t)}{b+R(t)} - \frac{H^*(t)}{b+R(t)}\right)sign(R(t)-R^*(t)) \\ &\quad + \frac{e+a}{b+R(t)}(R(t)-R^*(t))sign(R(t)-R^*(t)) \\ &\le -\frac{r}{K}|R(t)-R^*(t)| - \frac{aK}{b}|H(t)-H^*(t)| + \frac{\eta}{b+R(t)}|R(t)-R^*(t)| \\ &\le -\left|\frac{r}{K} - \frac{e+a}{b}\right| |R(t)-R^*| - \frac{aK}{b}|H(t)-H^*(t)|. \end{aligned}$$

Therefore

$$D^+V(t,R(t),H(t)) \le -\left|\frac{r}{K} - \frac{\eta}{b}\right| |R(t)-R^*(t)| - \frac{aK}{b}|H(t)-H^*(t)|. \qquad (2.30)$$

From (2.28) and (2.30) we obtain

$$D^+V(t,R(t),H(t)) \le -\left|\frac{r}{K} - \frac{\eta}{b}\right| M_R \left|\ln\frac{R(t)}{R^*(t)}\right| - \frac{aK}{b} M_H \left|\ln\frac{H(t)}{H^*(t)}\right|.$$

Let

$$m_0 = min\left(\left|\frac{r}{K} - \frac{\eta}{b}\right| M_R, \frac{aK}{b}M_H\right)$$

then

$$D^+V(t,R(t),H(t)) \le -m_0V(t,R(t),H(t)). \qquad (2.31)$$

(2.29) and (2.31) give the following impulsive differential inequation:

$$\begin{cases} D^+V(t,R(t),H(t)) \le -m_0V(t,R(t),H(t)) \\ V^+\left(t_k^+,R(t_k^+),H(t_k^+)\right) \le V\left(t_k,R(t_k),H(t_k)\right). \end{cases} \qquad (2.32)$$

Using the theory of impulsive differential inequalities in [20] we have:

$$\begin{aligned} V(t,R(t),H(t)) &\le V\left(t_0^+,R(t_0^+),H(t_0^+)\right)e^{-m_0(t-t_0)} \\ &\le V\left(t_0^+,R(t_0),H(t_0)\right)e^{-m_0(t-t_0)} \end{aligned}$$

$$V(t,R(t),H(t)) \le V\left(t_0,R(t_0),H(t_0)\right)e^{-m_0(t-t_0)}. \qquad (2.33)$$

To show the uniqueness of the positive solution, we proceed as Tamen and Li in [13], [11] respectively. Consider the operator

$$\begin{aligned} \Psi: \mathbb{R}^2 &\mapsto \mathbb{R}^2 \\ \bar{X} &\mapsto \bar{X}(s+t_k,t_0,\bar{R}_0) \end{aligned}$$

$s \in [t_0,+\infty[$, where $\bar{X} = (\bar{R}(t),\bar{H}(t))$, , is a positive solution of (2.34). For $\bar{X}_0 = (\bar{R}_0,\bar{H}_0)$, we deduce that:

$\Psi^k\bar{R}_0 = \bar{X}(s+kt_k,t_0,\bar{R}_0)$. (2.34)

Suppose there exists $a, b \in K$ such that $a(\|X-\bar{X}\|) \le V(t,X) \le b(\|X-\bar{X}\|)$). From (2.34), we have :

$$\|R(t)-\bar{R}(t)\|_\infty + \|H(t)-\bar{H}(t)\|_\infty \le a^{-1}(b\|R_0-\bar{R}\|_\infty + \|H_0-\bar{H}_0\|_\infty)e^{-m_0(t-t_0)}$$

Since $e^{-m(t-t_0)} \to 0$ when $t \to \infty$, then there exists $\Phi > t_0$ such that for all $0 < \varepsilon < 1$,

$\|(R(t),H(t)) - (\bar{R},\bar{H})\| \le \frac{\varepsilon}{2}\|(R_0,H_0) - (\bar{R}_0,\bar{H}_0)\|, \forall t > \Phi$. (2.35)

By choosing $k = \frac{[\Phi]+1}{\tau}$ in (2.35) we have $K\tau = [\Phi]+1 > \Phi$ then (2.35) and (2.36) imply

$$\begin{aligned} \left\|\Psi^kR_0 - \Psi^k\bar{R}_0\right\| &= \|(R,H)(s+[\Phi]+1,t_0,R_0) - (\bar{R},\bar{H})(s+[\Phi]+1,t_0,R_0)\| \\ &\le \frac{\varepsilon}{2}\|X_0 - \bar{X}_0\|. \end{aligned}$$

So Ψ is a contracting operator in the Banach space $\mathbb{R}^2$. Therefore, according to the Banach fixed point theorem, there exists a unique $\mathbf{X}^* = (\mathbf{R}^*,\mathbf{H}^*) \in \mathbb{R}^2$ such that $\mathbf{\Psi X}^* = \mathbf{X}^*$.

Hence

$$\begin{cases} \mathbf{R}(\mathbf{s}+\mathbf{t}_\mathbf{k},\mathbf{t}_0,\mathbf{R}^*) = \mathbf{R}^* \\ \mathbf{H}(\mathbf{s}+\mathbf{t}_\mathbf{k},\mathbf{t}_0,\mathbf{H}_0^*) = \mathbf{H}^* \end{cases} \qquad (2.36)$$

where $(\mathbf{R},\mathbf{H})$ is a positive solution of (1.2). Using the theorem (existence of at least one solution) which gives the sufficient condition that guarantees the existence of at least one positive τ-periodic solution $(\mathbf{R},\mathbf{H})$. We conclude that there is a unique τ-periodic solution of (1.2).

Taking intoaccount(2.28)and(2.33),werealizefromtheabovethat alltheassumptions of the theorem (uniform asymptotic stability)aresatisfied. It followsthat the τ-periodicladybug pulceron(R,H)solutionofsystem(1.2)isuniformlyasymptotically stable.

Sinceinequation(2.36)impliesthatV(t,x(t),y(t))→0(t→ ∞), thenR(t)→ R^*(t)and

$H(t) \rightarrow H^*(t)$whent$\rightarrow \infty$. Asaresult, the τ-periodicresource-herbivore solution isgloballystable.

■

3. Numerical Simulations

In this section, we present some numerical simulations to illustrate our theoretical results presented in Section 2. Parameter values have been chosen to respect the existence and stability conditions of the model equilibria.

Figure 1 shows the stability of equilibrium $\mathbf{E_0} = (\mathbf{0}, \mathbf{0})$. This stability of equilibrium $\mathbf{E_0}$means that there will be no more resources or livestock in the area under consideration for a long time. There are many possible explanations for this situation. Droughts can lead to the death of livestock through starvation, emergency slaughter, sale or permanent herd migration (transhumance).

Figure 2 and Figure 3 (A)-(B) shows the stability of coexistence equilibrium $\mathbf{E}^* = (\mathbf{R}^*, \mathbf{H}^*)$. This means that, under certain conditions, the resource and herbivores can coexist for a long time in the area under consideration. This situation would prevent herders from transhumance.

Figure 4shows that impulse has no impact on resource dynamics: the more cattle are harvested (with$\boldsymbol{\tau} = \mathbf{5}$), the more the number of herbivores decreases and the resource increases.

Figures 5 A with a period$\tau = 6$; $t_{max} = 100$ and 5 A with a period $\tau = 8$; $t_{max} = 262$ show the stability of the coexistence equillibrium in the model with impulse harvesting. This coexistence between the resource and the herbivores shows an oscillatory equilibrium behavior. An increase in the resource is followed by an increase in the number of herbivores, and a decrease in the resource is followed by a decrease in the number of herbivores. During this resource depletion, herders harvest livestock for sale, bequeathing or emergency slaughter. This harvesting is carried out at very specific times, enabling better management of herbivore numbers.

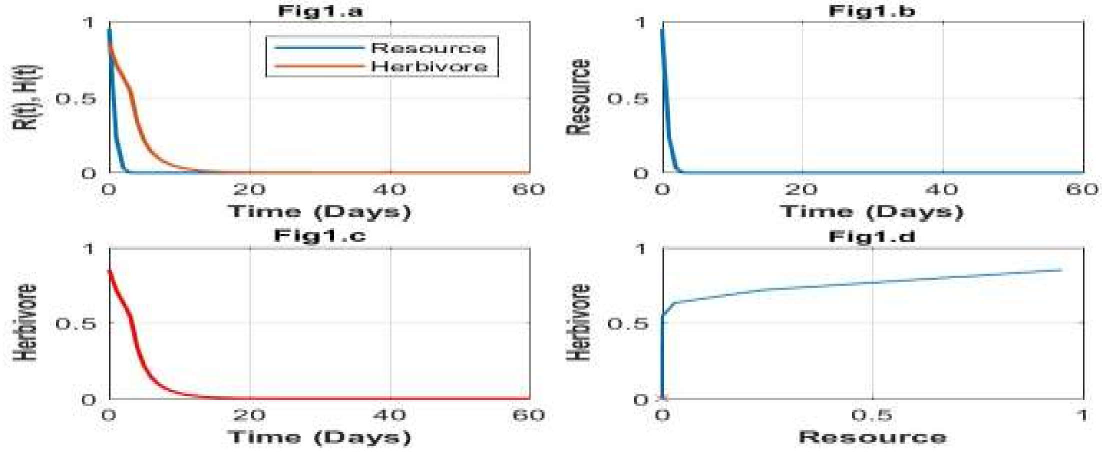


Figure 1:Chronicleofresourcesand herbivores asafunctionoftimeat the equilibrium point(0,0),withparameters$r=-0.75$,$a=0.99$;$\mu=0.3$;$c=0.43$;$b=0.006$;$K=100$;$e=0.9$; $h=0.06$

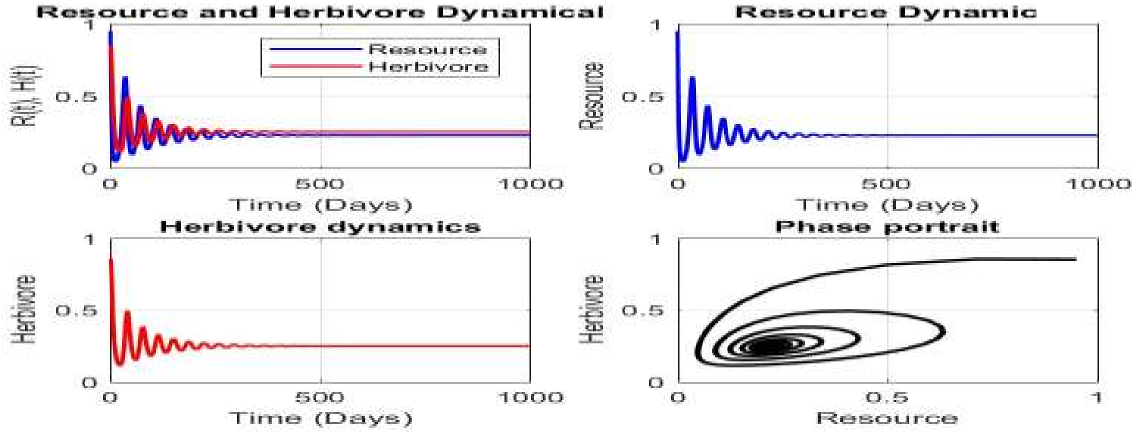
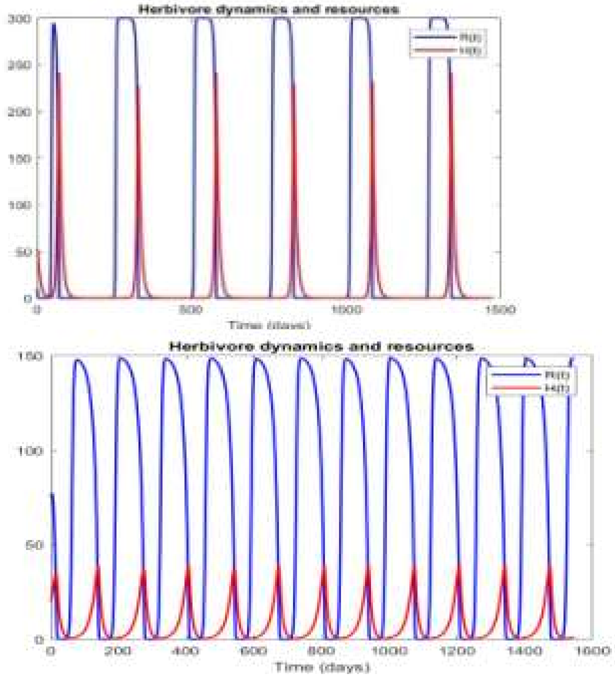


Figure 2:Chronicles ofResource and Herbivore densities, with parameters:$r = 0.75$;$a=0.62$;$\mu=0.3$;$c=0.43$;$b=0.22$;$K=100$;$e=0.5$;$h=0.92$;$R_0=0.95$;$H_0=0.85$.



(A) (B)

Figure3:r=0.85;k=300;a=0.74;b=10;c=0.5;e=0.5;μ=0.1;h=0.1

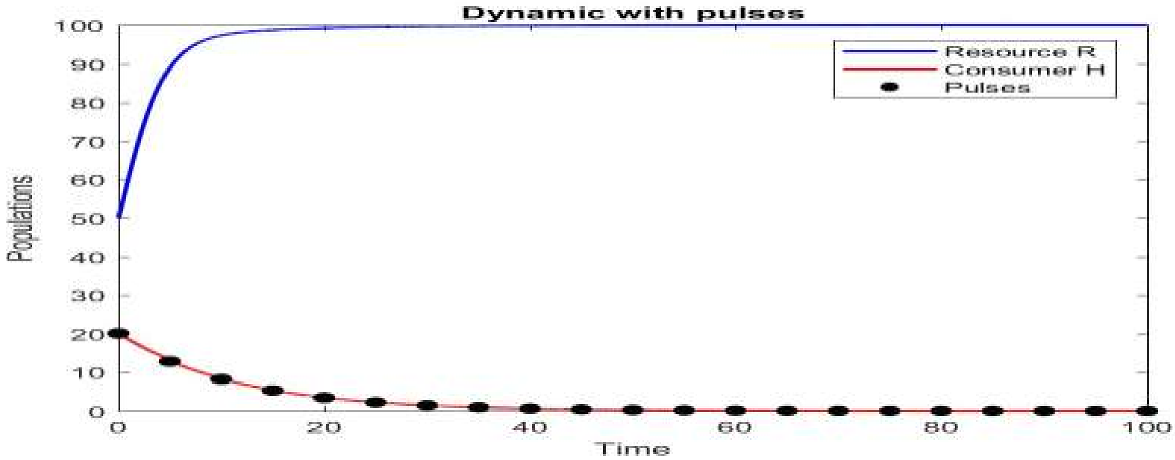
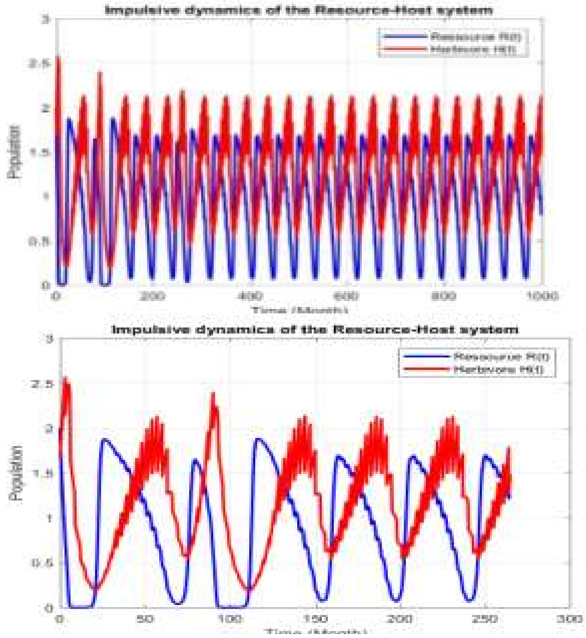


Figure4:r=0.85;k=300;a=0.74;b=10;c=0.5;e=0.5;μ=0.1;h=0.1; $\tau = 5$.



(A) (B)

Figure 5:r = 0.5; K = 100; a = 0.1; b = 10; e = 0.2; μ = 0.1; h = 0.05; c = 5.

3. Conclusion and Perspectives.

In this article, we have developed and analyzed a herd-resource interaction model. The aim is to study a plant-herbivore interaction model in a Sahelian region. In addition, we have taken into account the continuous and impulse harvesting of the herbivore population, which is motivated by empirical evidence in Sahelian regions. Indeed, in these regions of Africa, the herd is exploited mainly for survival needs (food, sale, bequest etc.). The theoretical and numerical analysis of our model reveals various equilibria and rich dynamics.Previous studies have shown the great importance of pastoralism in the national economy of arid tropical countries. However, it is confronted by two major difficulties, namely low rainfall levels in the area caused by climate change and massive harvesting of livestock for population survival.Mathematical analysis has shown that resources and herbivores can disappear. This may be due to long droughts (i.e.$r < 0$).This phenomenon leads farmers to migrate with their livestock (transhumance). In addition, as certain parameters of the system change, several bifurcations appear, notably the Hopf bifurcation. In the case of impulsive harvesting, we have shown that the resource-free solution is τ-periodic and locally asymptotically stable and the existence of the τ-periodic resource-herbivore solution that we have shown to be stable using the theory of impulse equations.Due to the unavailability of parameter values in the Sahelian region it is desirable that some field experiments can be conducted to assess at least the magnitude of (some) parameters. For example, assessing the magnitude or range of the half-saturation parameter b and the rate of conversion of resource biomass to herbivore biomass e, which are two bifurcation parameters of the model.In the following, we can take into account the temporal and spatial behavior of herbivores and their interactions with resources by using a local spatial operator (Laplace operator) and not local with certain kernels. We can also consider several sites and look for the best optimal strategy to achieve both sustainable resource management in different locations and good herbivore feeding.

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