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RESEARCH ARTICLE

ON THE HOMOMORPHISM OF FINITE ABELIAN GROUPS

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Abstract

In this article, we considered the group $Hom(A, B)$of the homomorphisms of two finite abelian groups A and B. We studied the conditions in which the group $Hom(A, B)$is isomorphic to Aor B for all finite abelian groups A and B.

Introduction:-

This article is placed within the framework of the study of the group $Hom(A, B)$of the homomorphisms of two abelian groups A and Bwhich constitutes the problem 30 of Laslo FUCHS posed in 1977. Let us consider that in the group $Hom(A, B)$, the abelian groups A and $B$are finite abelian groups, then we obtain a special case of the problem 30 of Laslo FUCHS.

We consider the group $Hom(A, B)$of the homomorphisms of two finite abelian groups, we show that it is equal to $\{0\}$ if and only if the cardinals of groups A and B are coprime and the group $Hom(A, B) \cong A$ if and only if A is cyclic. In particular, if A and B are cyclic, we show that the group $Hom(A, B)$is isomorphic to B if and only if the cardinality of B divides that of A for all cyclic groups $A$and B.

We will note by:
- $Z/n\,Z$the cyclic group of order n;
- $A_p$the p-cyclic subgroup whose order is a power of$p$;
- $Hom(A, B)$: the group of the homomorphisms from group A to group B;
- $P(A) = \{p \in P : A_p \neq 0\}$;
- $o(a)$:the order of an element $a \in A$;
- $\oplus$: the direct sum
- |A|: the cardinal of group A;
- N:the set of natural numbers;
- P:the set of prime natural numbers;
- Kerφ: the kernel of the homomorphism φ;
- S_3:the symmetric group of order 3;

The other notations are taken from [5] or introduced along the way.

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2. **Problem**:
In works [3] and [4], we posed the following problems:
a) When is $Hom(A,B)$ isomorphic to A or B?
b) Describe all abelian groups A such that $Hom(A,A) \cong A$.
3. Results and discussions
For two finite abelian groups A and B, we describe the following results.
Theorem 1. Let A be any group and B be an abelian group whose law is addition.
Then, $(Hom(A,B),+)$ is an abelian group.
Proof. Indeed, let us define in $(Hom(A,B),+)$ the following operation:
$(\varphi+\beta)(a)=\varphi(a)+\beta(a), \forall a \in A$.
The operation + thus defined in $(Hom(A,B),+)$ is associative and commutative. Any homomorphism φ of $Hom(A,B)$ admits an opposite $-\varphi$ of $Hom(A,B)$ such that $(-\varphi)(a)=-\varphi(a) \forall a \in A$.
There exists the zero homomorphism $e_{Hom(A,B)}=\varphi_0 \in Hom(A,B)$ such that $(\varphi+\varphi_0)(a)=\varphi(a)+\varphi_0(a)=\varphi(a), \forall a \in A$ because $\varphi_0(a)=e_B$.■
Lemma 1. [5] Let A be a finite abelian group of order $n=p_1^{\alpha_1}.p_2^{\alpha_2}\dots p_k^{\alpha_k}$. Then A is a direct sum of its p_i-cyclic subgroups whose order is a power of p_i, $(i=1,2,\dots,k)$. ■
Theorem 2. Let A and B be two finite abelian groups of order n and m respectively.
Then, $Hom(A,B)=\{0\}$ if and only if n and m are coprime.
Proof. Indeed, by Lemma, we can note $A=A_{p_1}\oplus\dots\oplus A_{p_k}$,
$\forall p_i \in P(A)$ and $B=B_{p_1}\oplus\dots\oplus B_{p_t}, \forall p_j \in P(B)$ with $P(A)=\{p_i \in P: A_{p_i}\neq 0\}$ where A_{p_i} are p_i −cyclic subgroups whose order is a power of $p_i (i=1,2,\dots,k)$, $P(B)=\{p_j \in P: B_{p_j}\neq 0\}$ where B_{p_j} are p_j −cyclic subgroups whose order is a power of p_j $(j=1,2,\dots,t)$.
Note that $|A_{p_1}|=p_1^{\alpha_1};\dots;|A_{p_k}|=p_k^{\alpha_k}$ and $|B_{p_1}|=p_1^{\beta_1};\dots;|B_{p_t}|=p_t^{\beta_t}$, which assures us of the fact that

$$Hom(A,B)\cong Hom(A_{p_1}\oplus\dots\oplus A_{p_k}, B_{p_1}\oplus\dots\oplus B_{p_t})\cong \bigoplus_{i\in 1,k}\bigoplus_{j\in 1,t} Hom\left(A_{p_i},B_{p_j}\right).$$

So, $Hom(A,B)=\{0\}\Leftrightarrow \bigoplus_{i\in 1,k}\bigoplus_{j\in 1,t} Hom\left(A_{p_i},B_{p_j}\right)=\{0\}, \forall p_i \in P(A), \forall p_j \in P(B) \Leftrightarrow\Leftrightarrow Hom\left(A_{p_i},B_{p_j}\right)=\{0\}\forall p_i,p_j \in P(A)\cap P(B) \Leftrightarrow P(A)\cap P(B)=\emptyset \Leftrightarrow (m,n)=1$.
Hence the integer n is coprime with m. ■
Corollary1. For any finite abelian group A of order $p\in P$, there always exists a finite abelian group $B\neq\{0\}$ of order different from p such that $Hom(A,B)=\{0\}$.
Proof. Indeed, let B be a finite abelian group of order $n=p_1^{\alpha_1}.p_2^{\alpha_2}\dots p_k^{\alpha_k}$. Then the group $B=B_{p_1}\oplus\dots\oplus B_{p_k}$ for all $p_i \in P(B)$ where B_{p_i} are p_i-cyclic subgroups whose order is a power of $p_i (i=1,2,\dots,k)$ and $|B_{p_i}|=p_i^{\alpha_i}\neq p$. It follows from this fact that
$Hom(A,B)\cong Hom(A,B_{p_1}\oplus\dots\oplus B_{p_k})\cong Hom(A,B_{p_1})\oplus\dots\oplus Hom(A,B_{p_k})=\{0\}$ by application of theorem1. ■
Corollary 2. Let A and B be two finite abelian groups of order p and q respectively such that $p\neq q$ with $p,q\in P$.
Then, $Hom(A,B)=\{0\}$.■
Proposition. Let A and B be two cyclic abelian groups of order n and m respectively.
Then, $Hom(A,B)\cong B$ if and only if m divides n.
Proof. Indeed, as A and B are cyclic then $A\cong Z/nZ$ and $B\cong Z/mZ$. Thus we obtain: $Hom(A,B)\cong B \Leftrightarrow Hom(Z/nZ, Z/mZ)\cong Z/mZ \Leftrightarrow m$ divides n. ■
Examples.
1) $A=Z/2^4Z\oplus Z/3^2Z$ and $B=Z/4Z\oplus Z/3Z$, we have: $Hom(A,B)\cong B$.
2) $A=Z/3^4Z\oplus Z/5Z$ and $B=Z/2Z\oplus Z/2Z$, we have: $Hom(A,B)=\{0\}$.
Theorem 3. If $A=S_3$ the symmetric group of order 3 and $B=Z/2Z$ the cyclic group of order 2, then $Hom(A,B)\cong B$.
Proof. Indeed, let $A=S_3=\{\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6\}$ such that $\beta_1=(123)$, $\beta_2=(213)$,
$\beta_3=(321)$, $\beta_4=(132)$, $\beta_5=(231)$, $\beta_6=(312)$ and $B=Z/2Z=\{\dot{0},\dot{1}\}$.
Then, $e_{Hom(A,B)}=\varphi_0\in Hom(A,B)$ and Kerφ_0=S_3. Consider a homomorphism $\varphi_1\in Hom(A,B)$ where $\varphi_1(\beta_1)=0$, $\varphi_1(\beta_5)=0$, $\varphi_1(\beta_6)=0$ with $\beta_5(1)=2, \beta_5(2)=3, \beta_5(3)=1, \beta_6(1)=3, \beta_6(2)=1, \beta_6(3)=2$. So, Ker$\varphi_1$=$\{\beta_1,\beta_5,\beta_6\}$.

For any non-zero homomorphism φ ($\varphi \neq \varphi_1$), let Kerφ={β_1} but this is impossible.
Hence $Hom(A,B) = \{\varphi_0, \varphi_1\}$ and consequently, $Hom(A,B) \cong B$.■
Theorem 4. Let A be a finite abelian group. Then the following statements are equivalent:
a) A is cyclic
b) A_{p_i} is cyclic for all $p_i \in P(A)$
c) $Hom(A,A) \cong A$.
Proof. a) ⇒ b). Indeed, if A is cyclic, then $A = A_{p_1} \oplus \dots \oplus A_{p_k}$ where A_{p_i} are cyclic groups for all $i \in \{1,2,\dots,k\}$ and $p_i \in P(A) = \{p_i \in P: A_{p_i} \neq 0\}$ because any subgroup of a cyclic group is cyclic.
b) ⇒ c). Note that $|A_{p_1}| = p_1^{\alpha_1}; \dots; |A_{p_k}| = p_k^{\alpha_k}$, which assure fact that
$Hom(A,A) \cong Hom(A_{p_1} \oplus \dots \oplus A_{p_k}, A) \cong \bigoplus_{i\in 1,k} \bigoplus_{j \in 1,k} Hom\left(A_{p_i}, A_{p_j}\right)$.
Note that $Hom\left(A_{p_i}, A_{p_j}\right) \neq \{0\} \Leftrightarrow i = j$ and $o(a_i)$ divides $p_i^{\alpha_i}, \forall a_i \in A_{p_i}$ and $o(a_j)$ divides $p_j^{\alpha_j}, \forall a_j \in A_{p_j}$ but $o(a_j)$ does not divide $o(a_i)$.
So, $\varphi(a_i) = a_j \Rightarrow a_j = 0 \Rightarrow \varphi = 0 \Rightarrow Hom\left(A_{p_i}, A_{p_j}\right) = \{0\} (i \neq j!!!)$.
Now, let us show that $Hom(A_{p_i}, A_{p_i}) \cong A_{p_i}$.
Let $\varphi \in Hom(A_{p_i}, A_{p_i})$ such that $\varphi(a) = b, \forall a, b \in A_{p_i}$. It is clear that the equality $\varphi(a) = b$ gives a homomorphism $\varphi_1 \in Hom(A_{p_i}, A_{p_i})$ such that $\varphi_1(a_1) = b_1$ and $\varphi = \varphi_1 \Leftrightarrow b = b_1$. It follows that $|Hom(A_{p_i}, A_{p_i})| = |A_{p_i}| = p_i^{\alpha_i}$ and therefore $Hom(A_{p_i}, A_{p_i}) \cong A_{p_i} \neq \{0\}$.
Therefore,

$$Hom(A,A) \cong \bigoplus_{i\in 1,k} \bigoplus_{j \in 1,k} Hom\left(A_{p_i}, A_{p_j}\right) \cong \bigoplus_{i \in 1,k} Hom(A_{p_i}, A_{p_i}) \cong \bigoplus_{i \in 1,k} A_{p_i} \cong A.$$

c) ⇒ a). Supposing that
$Hom(A,A) \cong \bigoplus_{i\in 1,k} \bigoplus_{j \in 1,k} Hom\left(A_{p_i}, A_{p_j}\right) \cong \bigoplus_{i \in 1,k} Hom(A_{p_i}, A_{p_i}) \cong \bigoplus_{i \in 1,k} A_{p_i} \cong A$.But A being finite then, $A = A_{p_1} \oplus \dots \oplus A_{p_k}$ where A_{p_i} are cyclic for all $\in \{1,2,\dots,k\}$ and consequently, A is cyclic. ■
Corollary 3. For two cyclic abelian groups A and B, we have the following implication:

$$Hom(A,A) \cong Hom(B,B) \Rightarrow A \cong B.$$

Proof. Supposing that $Hom(A,A) \cong Hom(B,B)$ and by application of the previous theorem 4, we can affirm that $Hom(A,A) \cong A$ and $Hom(B,B) \cong B$. Which leads us to the required conclusion. ■
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Conclusion:-

In this work, we studied the group of the homomorphisms of finite abelian groups in general and that of cyclic groups in particular.

For two finite abelian groups A and B, we have shown that $Hom(A,B) = \{0\}$ if and only if their cardinals are relatively prime. If A and B are also cyclic, we show that $Hom(A,B) \cong B$ if and only if the cardinality of group B divides that of A.

We have also shown that, if the group A is arbitrary and B an abelian group, then the group $Hom\ (A,B)$ is abelian. In particular, we have established the case where $A = S_3$ and $B = Z/2Z$.

When is the group $Hom\ (A,B)$ isomorphic to the group B if B is an abelian group and A is any group?

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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