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RESEARCH ARTICLE

ALGEBRAIC EXTENSIONS THROUGH T-Q FERMATEAN $\mathcal{L}$-FUZZY IDEALS AND THEIR HOMOMORPHISMS

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Abstract

Fermatean fuzzy sets serve as a significant generalization of both intuitionistic fuzzy sets and Pythagorean fuzzy sets, providing a broader and more flexible structure for modeling uncertainty. Unlike their predecessors, they successfully address and overcome certain inherent limitations associated with these earlier frameworks, particularly in handling higher degrees of hesitation and indeterminacy. Motivated by these advantages, this paper introduces the concept of t-Q Fermatean L-fuzzy ideals, thereby extending the study of algebraic structures within the Fermatean fuzzy environment. We further explore the homomorphic properties of these ideals, analyzing how they behave under various mappings. Within this framework, a number of new theoretical results are established, which contribute to the deeper understanding of Fermatean fuzzy algebra and open avenues for further research.

Introduction:-

The foundation of fuzzy set theory was laid by Zadeh [23], who introduced the concept of a membership function ϱ to quantify the degree to which an element belongs to a given set. Unlike classical set theory, where membership is strictly binary an element either belongs to a set or it does not—fuzzy set theory allows for gradations of membership. Within this framework, every element of the universal set is assigned a membership value from the unit interval [0,1]. A value of 0 signifies complete non-membership, while a value of 1 indicates full membership. Intermediate values represent varying degrees of partial membership, capturing situations where the status of an element cannot be described in absolute terms. This innovative generalization of classical sets provides a powerful tool for modeling vagueness, uncertainty, and imprecision, since it reflects the reality that many real-world phenomena do not conform to rigid boundaries but instead fall within a spectrum of belonging.

Classical fuzzy set theory, despite its effectiveness in extending the binary nature of classical sets, exhibited notable limitations in its ability to model uncertainty in a comprehensive manner. Specifically, it lacked an explicit non-membership function to quantify the degree to which an element does not belong to a set, and it was unable to capture the hesitation or indeterminacy that often arises in real-world decision-making situations. Recognizing these

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shortcomings, Atanassov [9] proposed the concept of intuitionistic fuzzy sets (IFSs), which significantly enriched the fuzzy framework. An IFS is formally described by a triplet of functions: a membership function ϱ that assigns the degree of belonging of an element to a set, a non-membership function ϑ that expresses the degree of rejection, and an indeterminacy (or hesitation) function π that reflects the extent of uncertainty or lack of knowledge regarding the element's status. These functions are interrelated through the conditions $\varrho + \vartheta \leq 1$ and $\varrho + \vartheta + \pi = 1$, ensuring consistency in the representation of information. This formulation provides a richer and more flexible mechanism for representing vagueness and uncertainty, thereby broadening the applicability of fuzzy set theory in diverse fields such as decision-making, pattern recognition, and knowledge representation.

However, there are practical situations where the condition $\varrho + \vartheta \geq 1$ may hold, which is not permissible under IFSs. To accommodate such scenarios, Pythagorean fuzzy sets (PFSs) were introduced by Yager [21, 22]. In a PFS, the membership and non-membership degrees satisfy $0 \leq \varrho, \vartheta \leq 1$ with the constraint $\varrho^2 + \vartheta^2 \leq 1$, and the indeterminacy is derived accordingly as $\pi = \sqrt{1 - \varrho^2 - \vartheta^2}$. Fermatean fuzzy sets is the extension Pythagorean fuzzy sets. In fermatean fuzzy sets the membership grade (ϱ) and non-membership grade (ϑ) satisfy the conditions $0 \leq \varrho^3 + \vartheta^3 \leq 1$, where the values of ϱ and ϑ lie between 0 and 1.

In the context of algebraic structures, the study of fuzzy subsets in near-rings has a well-documented history. Kim and Jun [11] introduced the notion of intuitionistic fuzzification of various semigroup ideals. Later, Kyung Ho Kim and Young Bae Jun [12], in their work on "Normal fuzzy R-subgroups in near-rings", extended this line of study by defining normal fuzzy R-subgroups and investigating their properties. Kuncham et al. [13] subsequently introduced fuzzy prime ideals of near-rings. Further contributions include Solairaju and Nagarajan [19], who defined Q-fuzzy subrings, and Palaniappan, Arjanan, and Palanivelrajan [15], who introduced intuitionistic L-fuzzy subrings. Wang et al. [20] proposed intuitionistic fuzzy ideals of rings with threshold parameters (α, β), while Sharma [17] developed the concept of t-intuitionistic fuzzy quotient groups.

Building upon these foundational concepts, the present paper is devoted to the introduction and systematic study of t-Q Fermatean $\mathcal{L}$-fuzzy ideals. To provide a clear framework, the paper is organized as follows. Section 2 is dedicated to preliminaries, where we recall essential definitions and outline the key algebraic structures associated with Fermatean fuzzy sets and lattices, which form the basis for our study. Section 3 develops the central theme by formally introducing t-Q Fermatean $\mathcal{L}$-fuzzy ideals and investigating their fundamental properties, with particular emphasis on their behavior under homomorphisms. Finally, Section 4 concludes the work with a summary of the main findings and some closing observations that highlight the significance of the results and suggest possible directions for future research.

Preliminaries and Definition:-

We will review the related concepts of fuzzy sets, intuitionistic fuzzy sets, pythagorean fuzzy sets, fermatean fuzzy sets and lattices in this section.

Definition 2.1 We defined fuzzy set F in a universal set X as

$$F = \{\langle x, \varrho_F(x)\rangle : x \in X\},$$

where $\varrho_F : X \rightarrow [0,1]$ is a mapping that is known as the fuzzy membership function.

The complement of ϱ is defined by $\bar{\varrho}(x) = 1 - \varrho(x)$ for all $x \in X$ and denoted by $\bar{\varrho}$.

Definition 2.2 A fuzzy ideal ϱ of a ring R is called fuzzy primary ideal, if for all $a, b \in R$ either $\varrho(ab) = \varrho(a)$ or else $\varrho(ab) \leq (b^m)$ for some $m \in Z^+$.

Definition 2.3 A fuzzy ideal ϱ of a ring R is called fuzzy semiprimary ideal, if for all $a, b \in R$ either $\varrho(ab) \leq \varrho(a^n)$, for some $n \in Z^+$, or else $\varrho(ab) \leq (b^m)$ for some $m \in Z^+$

Definition 2.4 An intuitionistic fuzzy set (IFS) A in X is defined as

$$A = \{\langle x, \varrho_A(x), \vartheta_A(x)\rangle : x \in X\},$$

where the $\varrho_A(x)$ is the worth of membership and $\vartheta_A(x)$ is the worth of non-membership of the element $x \in X$ respectively.

Also $\varrho_A: X \to [0,1], \vartheta_A: X \to [0,1]$ and satisfy the condition

$$0 \leq \varrho_A(x) + \vartheta_A(x) \leq 1,$$

for all $x \in X$.

The degree of indeterminacy $h_A(x) = 1 - \varrho_A(x) - \vartheta_A(x)$.

Definition 2.5 A Pythagorean fuzzy set P in universe of discourse X is represented as

$$P = \{\langle x, \varrho_P(x), \vartheta_P(x)\rangle | x \in X\},$$

where $\varrho_P(x): X \to [0,1]$ denotes the worth of membership and $\vartheta_P(x): X \to [0,1]$ represents the worth to which the element $x \in X$ is not a member of the set P, with the condition that

$$0 \leq (\varrho_P(x))^2 + (\vartheta_P(x))^2 \leq 1,$$

for all $x \in X$.

The worth of indeterminacy $h_P(x) = \sqrt{1 - (\varrho_P(x))^2 - (\vartheta_P(x))^2}$.

Definition 2.6 A fermatean fuzzy set A in a finite universe of discourse X is furnished as

$$A = \{\langle x, \varrho_A(x), \vartheta_A(x)\rangle | x \in X\},$$

where $\varrho_A(x): X \to [0,1]$ denotes the worth of membership and $\vartheta_A(x): X \to [0,1]$ represents the worth to which the element $x \in X$ is not a member of the set A, with the predicament that

$$0 \leq (\varrho_A(x))^3 + (\vartheta_A(x))^3 \leq 1,$$

for all $x \in X$.

The worth of indeterminacy $h_A(x) = \sqrt[3]{1 - (\varrho_A(x))^3 - (\vartheta_A(x))^3}$.

Definition 2.7 Let X be a non empty set, and $\mathcal{L} = (\mathscr{L}, \leq)$ be a lattice with least element 0 and greatest element 1 and Q be a non empty set. A Q-L-fuzzy subset μ of X is a function $\mu: X \times Q \to \mathscr{L}$.

Definition 2.8 Let $\mathcal{L} = (\mathscr{L}, \leq)$ be a complete lattice with an evaluative order reversing operation $N: \mathscr{L} \to \mathscr{L}$ and Q be a non empty set.

Definition 2.9 A Q-Fermatean L-fuzzy subset (QFLFS) μ in X is defined as an object of the form $\mu = \{\langle (x,q), \varrho_\mu(x,q), \vartheta_\mu(x,q)\rangle : x \in X \text{ and } q \in Q\}$ where $\varrho_\mu: X \times Q \to \mathscr{L}$ and $\vartheta_\mu: X \times Q \to \mathscr{L}$ define the degree of member ship, and the degree of non membership of the element $x \in X$, respectively, and for every $x \in X$ and $q \in Q$.

Definition 2.10 Let R be a ring. A Q-Fermatean L-fuzzy subset μ of R is said to be a Q-Fermatean L-fuzzy sub ring (QFLFSR) of R if it satisfies the following axioms:

(i) $\varrho_\mu(x - y, q) \geq \min\{\varrho_\mu(x,q), \varrho_\mu(y,q)\}$

(ii) $\varrho_\mu(xy, q) \geq \min\{\varrho_\mu(x,q), \varrho_\mu(y,q)\}$

(iii) $\vartheta_\mu(x - y, q) \leq \max\{\vartheta_\mu(x,q), \vartheta_\mu(y,q)\}$

(iv) $\vartheta_\mu(xy, q) \leq \max\{\vartheta_\mu(x,q), \vartheta_\mu(y,q)\}$.

Definition 2.11 Let R be a ring. A Q-Fermatean L-fuzzy sub ring μ of R is said to be a Q-Fermatean L-fuzzy normal sub ring (QFLFNSR) of R if

(i) $\varrho_\mu(xy,q) = \varrho_\mu(yx,q)$

(ii) $\vartheta_\mu(xy,q) = \vartheta_\mu(yx,q)$ for all $x, y \in R$ and $q \in Q$.

Definition 2.12 Let μ be a QFLFS of a ring R. And let $t \in [0,1]$, then the μ^t of R is called the t-Q-Fermatean fuzzy subset (tQFLFS) of R with respect to (QFLFS) μ and is defined as $\mu^t = (\varrho_{\mu^t, \vartheta_{\mu^t}})$, where $\varrho_{\mu^t}(x,q) = min\{\varrho_\mu(x,q), t\}$ and $\vartheta_{\mu^t}(x,q) = max\{\vartheta_\mu(x,q), 1-t\}$, for all $x \in R$.

Definition 2.13 Let X, Y be two non empty sets and $\phi: X \to Y$ be a mapping. Let μ and γ be two tQFLFS of X and Y respectively. Then the image of μ under the map ϕ is denoted by $\phi(\mu)$ and is defined as $\phi(\mu^t)(y,q) = (\varrho_\phi(\mu^t)(y,q), \vartheta_\phi(\mu^t)(y,q))$, where

$$\varrho_\phi(\mu^t)(y,q) = \begin{cases} sup\{\varrho_{\mu^t}(x,q)\}, & x \in \phi^{-1}(y), \\ 0, & otherwise, \end{cases}$$

$$\vartheta_\phi(\mu^t)(y,q) = \begin{cases} inf\{\vartheta_{\mu^t}(x,q)\}, & x \in \phi^{-1}(y), \\ 1, & otherwise, \end{cases}$$

also the pre-image of γ under ϕ is denoted by $\phi^{-1}(\gamma^t)$ and is defined as

$$\phi^{-1}(\gamma^t(x,q) = (\varrho_{\phi^{-1}}(\gamma^t)(x,q), (\vartheta_{\phi^{-1}}(\gamma^t)(x,q)),$$

where $\varrho_{\phi^{-1}}(\gamma^t)(x,q) = \varrho_{\gamma^t}(\phi(x),q)$ and $\vartheta_{\phi^{-1}}(\gamma^t)(x,q) = \vartheta_{\gamma^t}(\phi(x),q)$.

This means that $\phi^{-1}(\gamma^t)(x,q) = (\varrho_{\gamma^t}(\phi(x),q), \vartheta_{\gamma^t}((\phi(x),q))$.

Definition 2.14 Let $\phi: X \to Y$ be a mapping.Let μ and γ be two tQFLFS of X and Y respectively. Then $\phi^{-1}(\gamma^t) = (\phi^{-1}(\gamma^t))^t$ and $\phi(\mu^t) = (\phi(\mu))^t$ for all $t \in [0,1]$.

Definition 2.15 Let μ be a QFLFS of a ring R. And let $t \in [0,1]$, then μ is called t-Q-Fermatean L-fuzzy sub ring (tQFLFSR) of R if is QFLFSR of R. This means that μ^t satisfies the following conditions:

1. $\varrho_{\mu^t}(x-y,q) \geq min\{\varrho_{\mu^t}(x,q), \varrho_{\mu^t}(y,q)\}$;

2. $\varrho_{\mu^t}(xy,q) \geq min\{\varrho_{\mu^t}(x,q), \varrho_{\mu^t}(y,q)\}$;

3. $\vartheta_{\mu^t}(x-y,q) \leq max\{\vartheta_{\mu^t}(x,q), \vartheta_{\mu^t}(y,q)\}$;

4. $\vartheta_{\mu^t}(x-y,q) \leq max\{\vartheta_{\mu^t}(x,q), \vartheta_{\mu^t}(y,q)\}$; for all $x, y \in R$ and $q \in Q$.

Theorem 2.1 If μ is QFLFNSR of a ring R, then μ is also tQFLFNSR of a ring R.

Proof. Let $x, y \in R$ be any elements, then

$$\varrho_{\mu^t}(xy,q) = min\{(xy,q), t\} = min\varrho_\mu, (yx,q), t = \varrho_{\mu^t}(yx,q).$$

Similarly, $\vartheta_{\mu^t}(xy,q) = max\{(xy,q), 1-t\} = max\{\varrho_\mu, (yx,q), 1-t\} = \vartheta_{\mu^t}(yx,q)$.

Therefore is also tQFLFNSR of R.

Definition 2.16 Let μ be a QFLFS of a ring R. And let $t \in [0,1]$, then μ is called t-Q-Fermatean L-fuzzy left ideal (tQFLFLI) of R. If

(i) $\varrho_{\mu^t}(x-y,q) \geq min\{\varrho_{\mu^t}(x,q), \varrho_{\mu^t}(y,q)\}$

(ii) $\varrho_{\mu^t}(xy,q) \geq \{\varrho_{\mu^t}(y,q)\}$

(iii) $\vartheta_{\mu^t}(x-y,q) \leq max\{\vartheta_{\mu^t}(x,q), \vartheta_{\mu^t}(y,q)\}$

(iv) $\vartheta_{\mu^t}(xy,q) \leq \{\vartheta_{\mu^t}(y,q)\}$ for all $y \in R$ and $q \in Q$.

Definition 2.17 Let μ be a QFLFS of a ring R. And let $t \in [0,1]$, then μ is called t-Q-Fermatean L-fuzzy right ideal (tQFLFRI) of R. If

(i) $\varrho_{\mu^t}(x-y,q) \geq min\{\varrho_{\mu^t}(x,q), \varrho_{\mu^t}(y,q)\}$

(ii) $\varrho_{\mu^t}(xy,q) \geq \{\varrho_{\mu^t}(x,q)\}$

(iii) $\vartheta_{\mu^t}(x-y,q) \leq max\{\vartheta_{\mu^t}(x,q), \vartheta_{\mu^t}(y,q)\}$

(iv) $\vartheta_{\mu^t}(xy,q) \leq \{\vartheta_{\mu^t}(x,q)\}$;

Theorem 2.2 If μ is QFLFLI of a ring R, then μ is also tQFLFLI of a ring R.

Proof. It is required to prove that $\varrho_{\mu^t}(xy,q) \geq \{\varrho_{\mu^t}(y,q)\}$ and $\vartheta_{\mu^t}(xy,q) \leq \{\vartheta_{\mu^t}(y,q)\}$ for all $x, y \in R$.

Again, $\varrho_{\mu^t}(xy,q) = min\{(xy,q),t\} = min\varrho_{\mu},(y,q),t = \varrho_{\mu^t}(y,q)$.

Thus $\varrho_{\mu^t}(xy,q) \geq \{\varrho_{\mu^t}(y,q)\}$. Similarly, we can show that $\vartheta_{\mu^t}(xy,q) \leq \{\vartheta_{\mu^t}(y,q)\}$.

Hence is also tQFLFLI of a ring R.

Definition 2.18 If μ is QFLFRI of a ring R, then μ is also tQFLFRI of a ring R.

Main Results

In this section, we have undertaken a detailed discussion of several significant results concerning the homomorphic behavior of t-Q Fermatean $\mathcal{L}$-fuzzy subrings. These results highlight how such structures interact under homomorphisms, providing deeper insights into their algebraic properties and contributing to a broader understanding of Fermatean fuzzy algebra within the framework of $\mathcal{L}$-fuzzy subrings.

Theorem 3.1 Let $\phi: R_1 \to R_2$ be a ring homomorphism from the ring R_1 into a ring R_2. Let γ be tQFLFSR of R_2**.** Then $\phi^{-1}(\gamma)$ is tQFLFSR of R_1**.**

Proof. Let $x, y \in R_1$, since γ be tQFLFSR of R_2**.** Then

$$\phi^{-1}(\gamma^t)(x-y,q) = (\varrho_{\phi^{-1}(\gamma^t)}(x-y,q), \vartheta_{\phi^{-1}(\gamma^t)}(x-y,q)).$$

$$\varrho^{-1}(\gamma^t)(x-y,q) = (\varrho_{\gamma^t}(\phi(x-y,q)$$

$$= \varrho_{\mu^t}(\phi(x)-\phi(y),q$$

$$\geq min\{\varrho_{\gamma^t}(\phi(x),q), \varrho_{\gamma^t}(\phi(y),q)\}$$

$$= min\{\varrho^{-1}(\gamma^t)(x,q), \varrho_{\phi^{-1}}(\gamma^t)(y,q).\}$$

Thus $\varrho^{-1}(\gamma^t)(x-y,q) \geq min\{\varrho^{-1}(\gamma^t)(x,q), \varrho_{\phi^{-1}}(\gamma^t)(y,q)\}$.

Similarly, it can be prove that $\vartheta_{\phi^{-1}(\gamma^t)}(x-y,q) \leq max\{\vartheta_{\phi^{-1}}(x-q), \vartheta_{\phi^{-1}}(y,q)\}$.

Again,

$$\vartheta_{\phi^{-1}(\gamma^t)}(x-y,q) = \varrho_{\gamma^t}(\phi(\text{xy}),q)$$

$$= \varrho_{\gamma^t}(\phi(x)\phi(y),q) \geq min\{\varrho_{\gamma^t(\phi(y),q)}\}$$

$$= min\{\varrho^{-1}(\gamma^t)(x,q), \varrho_{\phi^{-1}}(\gamma^t)(y,q).\}$$

Thus, $\varrho_{\phi^{(-1)(\gamma^t)}}(xy,q) \geq min\{\varrho^{-1}(\gamma^t)(x,q), \varrho_{\phi^{-1}}(\gamma^t)(y,q)\}$.

Also,

$$\vartheta_{\phi^{-1}(\gamma^t)}(xy,q) \leq max\{\vartheta_{\phi^{-1}}(x-q), \vartheta_{\phi^{-1}}(y,q)\}.$$

Therefore, $\phi^{-1}(\gamma^t) = (\phi^{-1}(\gamma))^t$ is QFLFSR of R_1 and hence $\phi^{-1}(\gamma^t)$ is tQFLFSR of R_1.

Theorem 3.2 Let $\phi: R_1 \to R_2$ be a ring homomorphism from the ring R_1 into a ring R_2. Let γ be tQFLFSR of R_2. Then $\phi^{-1}(\gamma)$is tQFLFSR of R_1.

Proof. Let $x, y \in R_1$, since γ be tQFLFSR of R_2. Also $\phi^{-1}(\gamma^t)(xy) = (\varrho_{\phi^{-1}(\gamma^t)}(xy,q),$

$\phi^{-1}(\gamma^t)(xy,q), \vartheta_{\phi^{-1}(\gamma^t)}(yx,q)$.

Hence, it is enough to show that

$$\varrho_{\phi^{-1}(\gamma^t)}(xy,q) = \varrho_{\phi^{-1}(\gamma^t)}(xy,q), \phi^{-1}(\gamma^t)(xy) \quad and \quad \vartheta_{\phi^{-1}(\gamma^t)}(xy,q) = \vartheta_{\phi^{-1}(\gamma^t)}(yx,q).$$

Now,

$$\varrho_{\phi^{-1}(\gamma^t)}(xy,q) = \varrho_{\gamma^t}(\phi(xy),q)$$

$$= \varrho_{\gamma^t}(\phi(x)\phi(y),q)$$

$$= \varrho_{\gamma^t}(\phi(y)\phi(x),q)$$

$$= \varrho_{\gamma^t}(\phi(xy),q)$$

$$= \varrho_{\phi^{-1}(\gamma^t)}(yx,q).$$

Moreover,

$$\vartheta_{\phi^{-1}(\gamma^t)}(xy,q) = \vartheta_{\gamma^t}(\phi(xy),q)$$

$$= \varrho_{\gamma^t}(\phi(x)\phi(y),q)$$

$$= \vartheta_{\gamma^t}(\phi(y)\phi(x),q)$$

$$= \vartheta_{\gamma^t}(\phi(xy),q)$$

$$= \vartheta_{\phi^{-1}(\gamma^t)}(yx,q).$$

Thus $\phi^{-1}(\gamma^t) = (\phi^{-1}(\gamma))^t$ is QFLFNSR of R_1 and hence $\phi^{-1}(\gamma^t)$ is tQFLFNSR of R_1.

Theorem 3.3 Let $\phi: R_1 \to R_2$ be a ring homomorphism from the ring R_1 into a ring R_2. Let γ be tQFLFLI of R_2. Then $\phi^{-1}(\gamma^t)$ is tQFLFLI of R_1.

Proof. Since γ be tQFLFSR of R_2 and let $x, y \in R_1$.

We need only to prove

$$\varrho_{\phi^{-1}(\gamma^t)}(xy, q) \leq \varrho_{\phi^{-1}(\gamma^t)}(y, q) \text{ and } \vartheta_{\phi^{-1}(\gamma^t)}(xy, q) \leq \vartheta_{\phi^{-1}(\gamma^t)}(y, q).$$

Now, $\varrho_{\phi^{-1}(\gamma^t)}(xy, q) = \varrho_{\gamma^t}(\phi(xy), q) = \varrho_{\gamma^t}(\phi(x)\phi(y), q) \geq \varrho_{\gamma^t}(\phi(y), q) = \varrho_{\phi^{-1}(\gamma^t)}(y, q)$.

Therefore, $\varrho_{\phi^{-1}(\gamma^t)}(xy, q) \geq \varrho_{\phi^{-1}(\gamma^t)}(xy, q)$

$$\varrho_{\phi^{-1}(\gamma^t)}(xy, q) = \varrho_{\gamma^t}(\phi(xy), q) = \varrho_{\gamma^t}(\phi(x)\phi(y), q) \geq \varrho_{\gamma^t}(\phi(y), q) = \varrho_{\phi^{-1}(\gamma^t)}(y, q).$$

Similarly, $\vartheta_{\phi^{-1}(\gamma^t)}(xy, q) \leq \vartheta_{\phi^{-1}(\gamma^t)}(y, q)$.

Therefore $\phi^{-1}(\gamma^t) = (\phi^{-1}(\gamma))^t$ is QFLFSR of R_1 and hence $\phi^{-1}(\gamma^t)$ is tQFLFSR of R_1.

Theorem 3.4 Let $\phi: R_1 \to R_2$ be a ring homomorphism from the ring R_1 into a ring R_2. Let γ be tQFLFRI of R_2. Then $\phi^{-1}(\gamma^t)$ is tQFLFRI of R_1.

Proof. Straight forward.

Theorem 3.5 Let $\phi: R_1 \to R_2$ be epimorphism from the ring R_1 into a ring R_2 and μ be tQFLFSR of R_1. Then $\phi(\mu)$ is tQFLFSR of R_2.

Proof. Let $x, y \in R_2$. Then there exist $a, b \in R_1$ such that $\phi(a) = x, \phi(b) = y$ we know that a, b need not be unique also μ is tQFLFSR of R_1.

Now, $\phi(\mu^t)(x - y, q) = (\varrho_{\phi(\mu^t)}(x - y, q), \vartheta_{\phi(\mu^t)}(x - y, q))$.

$$\varrho_{\phi(\mu^t)}(x - y, q) = \varrho_{(\phi(\mu))^t}(x - y, q) = min\{\varrho_{\phi(\mu)}(\phi(a) - \phi(b), q), t\}$$

$$\varrho^{-1}(\gamma^t)(x - y, q) \geq min\{\varrho^{-1}(\gamma^t)(x, q), \varrho_{\phi^{-1}}(\gamma^t)(y, q)\}.$$

Similarly, $\varrho_{\mu^{-1}(\gamma^t)}(x - y, q) \leq max\{\varrho_{\mu^{-1}}(x - q), \varrho_{\mu^{-1}}(y, q)\}$.

Also,

$$\vartheta_{\phi^{-1}(\gamma^t)}(x - y, q) = \varrho_{\gamma^t}(\phi(xy), q)$$

$$= \varrho_{\gamma^t}(\phi(x)\phi(y), q) \geq min\{\varrho_{\gamma^t(\mu(y),q)}\}$$

$$= min\{\varrho^{-1}(\gamma^t)(x, q), \varrho_{\phi^{-1}}(\gamma^t)(y, q).\}.$$

Thus, $\varrho_{\mu^{(-1)(\gamma^t)}}(xy, q) \geq min\{\varrho^{-1}(\gamma^t)(x, q), \varrho_{\phi^{-1}}(\gamma^t)(y, q)\}$.

It is easy to show that $\varrho_{\mu^{-1}(\gamma^t)}(xy, q) \leq max\{\varrho_{\mu^{-1}}(x - q), \vartheta_{\phi^{-1}}(y, q)\}$.

Similarly, we can show that

$$\vartheta_{\phi(\mu^t)}(x - y, q) \leq max\{\vartheta_{\phi(\mu^t)}(x, q), \vartheta_{\phi(\mu^t)}(y, q)\},$$

$$\varrho_{\phi(\mu^t)}(xy, q) = \varrho_{(\phi(\mu))^t}(xy, q) = min\{\vartheta_{\phi(\mu)}(\phi(a).\phi(b), q), t\}$$

$$= min\{\varrho_{\phi(\mu)}(\phi(ab), q), t\} \geq min\{\varrho_{\mu}(ab, q), t\} = \varrho_{\mu}(ab, q)'$$

for all $a, b \in R_1$ such that $\phi(a) = x, \phi(b) = y$.

$$= min\{sup\{\varrho_{\phi(\mu^t)}(a, q); \phi(a) = x\}, sup\{\varrho_{\phi(\mu^t)}(b, q); \phi(b) = y\}\}$$

$$= min\{\varrho_{\phi(\mu^t)}(x, q), \varrho_{\phi(\mu^t)}(y, q)\}.$$

Thus $\varrho_{\phi(\mu^t)}(xy, q) \geq min\{\varrho_{\phi(\mu^t)}(x, q), \varrho_{\phi(\mu^t)}(y, q)\}$.

Similarly, we can show that $min\{\varrho_{\phi(\mu^t)}(x, q), \varrho_{\phi(\mu^t)}(y, q)\}$.

Thus $\phi(\mu^t) = (\phi(\mu^t))'t$ is QFLFSR of R_2 and hence $\phi(\mu)$ is tQFLFSR of R_2.

Theorem 3.6 Let $\phi: R_1 \to R_2$ be epimorphism from the ring R_1 into a ring R_2 and μ be tQFLFNSR of R_1. Then $\phi(\mu)$ tQFLFNSR of R_2.

Proof. Let $x, y \in R_2$. Then exist $a, b \in R_1$ such that $\phi(a) = x, \phi(b) = y$ we know that a, b need not be unique also μ is tQFLFNSR of R_1. $\phi(\mu^t)(xy, q) = \varrho_{\phi(\mu^t)}(xy, q), \vartheta_{\phi(\mu^t)}(xy, q))$. Now, we have to prove that $\varrho_{\phi(\mu^t)}(xy, q) = \varrho_{\phi(\mu^t)}(yx, q)$ and $\vartheta_{\phi(\mu^t)}(xy, q) = \vartheta_{\phi(\mu^t)}(yx, q)$;

$$\varrho_{\phi(\mu^t)}(xy, q) = \varrho_{\phi(\mu^t)}(\phi(a)\phi(b), q)$$

$$= \varrho_{\phi(\mu^t)}(\phi(ab), q)$$

$$= sup\{\varrho_{\phi(\mu^t)}(xy, q); \phi(ab) = xy\}$$

$$= sup\{\varrho_{\phi(\mu^t)}(yx, q); \phi(ab) = xy\}$$

$$= \varrho_{\phi(\mu^t)}(\phi(ab), q)$$

$$= \varrho_{\phi(\mu^t)}(\phi(a)\phi(b), q)$$

$$= \varrho_{\phi(\mu^t)}(yx, q)$$

Similarly, we can show that $\vartheta(xy, q) = \vartheta(yx, q)$;.

Hence the result.

Theorem 3.7 Let $\phi: R_1 \to R_2$ be epimorphism from the ring R_1 into a ring R_2 and μ be tQFLFLI of R_1. Then $\phi(\mu)$ is tQFLFLI of R_2.

Proof. Let $x, y \in R_2$. Then there exist $a, b \in R_2$, then there exist a unique a,be R_1 such that $\phi(a) = x, \phi(b) = y$,

$$(\phi(\mu))^t(xy, q) = (\varrho_{(\phi(\mu))})^t(xy, q), (\varrho_{(\phi(\mu))})^t(xy, q)).$$

Since μ be IQFLFLI of R_1, then $\vartheta_{\phi(\mu^t)}(y, q) \geq \vartheta_{\phi(\mu^t)}(y, q)$ and

therefore $\varrho_{(}\phi(\mu))^t(xy, q) \geq \varrho_{(}\phi(\mu))^t(xy, q))$.

Similarly, it can be shown that

$$\vartheta_{(}\phi(\mu))^t(xy, q), \leq \vartheta_{(}\phi(\mu))^t(xy, q)).$$

Hence (μ^t) is QFLFLI of R_2 and hence $\phi(\mu)$ is tQFLFLI of R_2.

Theorem 3.8 Let R_1, R_2 be any two rings. The homomorphic image of a tQFLFSR of $\phi(R_1)$ is a tQFLFSR of $\phi(R_1) = R_2$.

Proof. Let μ be a tQFLFSR of R_1. We have to prove that γ is tQFLFSR of R_2.
Now, for $\phi(x)$, $\phi(y) \in R_2$ and $q \in Q$.

$$\varrho_{\gamma^t}(\phi(x) - \phi(y), q) = \varrho_{\gamma^t}(\phi(x - y), q) = min\varrho_\gamma\{(\phi(x - y), q), t\}$$

$$\geq min\varrho_\gamma\{\phi(x - y, q), t\} = min\{\varrho_{\gamma^t}(\phi(x), q), \varrho_{\gamma^t}(\phi(y), q)\}.$$

Also, for $\phi(x), \phi(y) \in R_2$ and $q \in Q$,

$$\varrho_{\gamma^t}(\phi(x)\phi(y), q) = \varrho_{\gamma^t}(\phi(xy), q) = min\{\varrho_\gamma(\phi(xy), q), t\}$$

$$\geq min\{\varrho_\gamma(xy, q), t\} = min\{\varrho_{\gamma^t}(x, q), \varrho_{\gamma^t}(y, q)\}.$$

Thus, $\varrho_{\gamma^t}(\phi(x)\phi(y), q) \geq min\{\varrho_{\gamma^t}(\phi(x, q), \phi(y, q))\}$.

Similarly, in can be prove that

$\vartheta_{\gamma^t}(\phi(x) - \phi(y), q) \leq max\{\vartheta_{\gamma^t}(\phi(x), q), \vartheta_{\gamma^t}(\phi(y), q)\}$ and

$\vartheta_{\gamma^t}(\phi(x)\phi(y), q) \leq max\{\vartheta_{\gamma^t}(\phi(x, q), \phi(y, q))\}$.

Hence γ is a tQFLFSR of R_2.

Theorem 3.9 Let R_1, R_2 be any two rings. The homomorphic image of a tQFLFNSR of R_1 is a tQFLFNSR of $\phi(R_1) = R_2$.

Proof. Since μ is a tQFLFSR of R_1. We have to prove that γ is a tQFLFSR of R_2.

Now for $\phi(x)$, $\phi(y) \in R_2$ and $q \in Q$, clearly γ is tQFLFSR of R_2.

Also, μ is tQFLFSR of R_1.

Again, $\varrho_{\gamma^t}(\phi(x)\phi(y), q) = \varrho_{\gamma^t}(\phi(xy), q) \geq \varrho_{\mu^t}(xy, q)$

$= \varrho_{\mu^t}(yx, q) = \varrho_{\mu^t}(\phi(yx), q) = \varrho_{\mu^t}(\phi(y)\phi(x), q)$.

Thus, $\varrho_{\gamma^t}(\phi(x)\phi(y), q) = \varrho_{\gamma^t}(\phi(y)\phi(x), q)$ for all $\phi(x)$, $\phi(y) \in R_2$ and $q \in Q$.

Also, $\vartheta_{\gamma^t}(\phi(x)\phi(y), q) = \vartheta_{\gamma^t}(\phi(xy), q) \leq \vartheta_{\mu^t}(xy, q) = \vartheta_{\mu^t}(yx, q) = \vartheta_{\gamma^t}(\phi(y)\phi(\mathrm{x}), q)$.

Thus, $\vartheta_{\gamma^t}(\phi(x)\phi(y), q) = \vartheta_{\gamma^t}(\phi(y)\phi(x), q)$.

Therefore, γ is tQFLFSR of R_1.

Conclusion:-

In order to deal with cognitive uncertainty in a more comprehensive manner, Fermatean fuzzy sets have emerged as a powerful extension of intuitionistic fuzzy sets, offering greater flexibility in modeling hesitation and imprecision. Motivated by these advantages, this paper focuses on the study of t-Q Fermatean $\mathcal{L}$-fuzzy ideals in the context of normal rings. We introduce and investigate their structural characteristics, establishing several important properties related to their homomorphic behavior. These results not only enrich the theoretical foundation of Fermatean fuzzy algebra but also provide useful insights for further applications. Looking ahead, a promising direction for future research lies in extending the framework to incorporate the concept of rough Fermatean fuzzy sets. In particular, we aim to develop and prove a number of significant theorems concerning rough Fermatean fuzzy sets in rings, which would further enhance the applicability of this theory in handling uncertainty and approximation in algebraic systems.

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