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RESEARCH ARTICLE

UNIFYING INFLATION-DARK ENERGY THROUGH SCALAR FIELD AND QUADRATIC TELEPARALLEL MODEL CONSTRAINED BY OBSERVATIONAL DATA

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Abstract

The present work investigates two of the most persistent and challenging phases in the dynamical evolution of the Universe. A quadratic teleparallel T^2gravity model, coupled to a scalar field assumed to be the sole constituent of the Universe, is constrained using observational data in order to derive meaningful results concerning both the inflationary era and the late-time accelerated expansion driven by dark energy. The Friedmann-like equations arising from the quadratic model are solved using the method of separation of variables. Firstly, explicit expressions for the scalar field and its potential are derived, allowing the computation of inflationary observables, including the number of e-folds analysis. For appropriate choices of the model parameters, a numerical analysis yields results consistent with the latest Planck observations and the BICEP2 experiment. Secondly, the energy density and the pressure of the scalar field are expressed as functions of the redshift z, within a framework where the Hubble parameter-obtained from the resolution of the Friedmann-like equations is constrained by current observational data. Under these conditions, the equation-of-state parameter$\omega_{\phi}(z)$tends toward -1, leading to the conclusion that the scalar field effectively mimics a ΛCDM like behaviour when the quadratic teleparallel model is confronted with observational constraints.

Introduction:-

Several cosmological investigations, especially cosmological observation have given evidence of the current accelerated expansion of the universe [1]-[4]. Such a scenario is powered by the so-called dark energy, whose widely accepted candidate for its explanation is the cosmological constant [5]. This choice is supported by observational approach promoted by the standard model of cosmology. When theoretically searching for the nature of dark energy, several approaches have been introduced with the goal of confirming observational predictions. which we can cite quintessence,

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K-essence, tachyon, phantom and dilatonic models. Under these approaches, it is hoped that an equation of state which varies with time, with the possibility of crossing the phantom barrier ($\omega = -1$), can be obtained [9]-[11]. Furthermore, the multi-component nature of dark energy, including the cosmological constant and scalar field, is also explored in these works and is shown to fit more observational data.

Moreover, other sources sustain the idea that dark energy may have a geometrical origin, i.e., that there is a connection between Dark Energy and a non-standard behavior of gravitation on cosmological scales, which has resulted in this topic becoming a very active area of research over the past few years (see for example [12]-[14]). The current acceleration of universe expansion has also benefited explanation based on the modification of the standard theory of gravitation: General Relativity, constructed on the non-vanishing curvature connexion, and its equivalent theory, called the teleparallel theory, based on non-vanishing torsion connexion [15]. The most addressed modified theories are $f(R)$ [16], $f(G)$ [17], $f(T)$ [18], etc... It is important to note here that, in addition to the dark energy problem for which they were initiated, these theories produce very interesting results in the study of inflation and also structures formation [19]-[23] in the universe. Especially, in the context of the description of inflation, the scalar field is added not only to provide a theoretical representation of the inflationary observable [24] and to compare them to observational data, but also to provide a gravitational Lagrangian density that mimics the same cosmological expansion as the scalar field-driven inflation of General Relativity. Recently, the introduction of scalar field in modified theories has further enriched the debates on the expansion of the universe, and especially on the exit from inflation [25]-[26].

The study of inflation and the ΛCDM model gives to the standard model of cosmology all the necessary tools to better reflect the realities of observational data. The challenge then lies in finding models that unify inflation and dark energy via the scalar field, as well as the transition between these phases. This is precisely the problem that this paper attempts to address partially. The problem will be addressed in the framework of $f(T)$ by initiating an unique scheme to explain together inflation and dark energy dominated era. As a brief motivation for this theory, the $f(T)$ theory leads to gravitational second-order field equations, the same as for GR, while it is of the fourth order in the context of $f(R)$. From this point of view, this theory is better adapted to deal with cosmological enigmas in modified theories of gravity. By the way, there exists one form of this theory which particularly retains a lot of attention: the quadratic $f(T)$ model, which is a formulation similar to the Starobinsky model in the $f(R)$ background [27], [28]. For example, the author in [29] has explained how the quadratic form of the scalar torsion can provide an origin for the late accelerated phase of the universe in the Friedmann-Robertson-Walker background. Furthermore, a gravitational model which can support simultaneously inflation and dark energy must be able to provide an exit from inflation.

A meaningful example can be seen in [30], where the trace-anomaly-driven inflation related to the quadratic model produces de Sitter inflation with a graceful exit. These works reveal the potential of the quadratic model in the dynamical study of the universe. Moreover, the interesting review on the $f(T)$ theory in [31] shows that there are few $f(T)$ models unifying inflation and dark energy. The few successful investigations are based on some approximations. Sometimes, only one of Friedmann's equations is used, which could lead to a loss of information. The present investigation will have the privilege of proposing a unified mechanism leading to the description of inflation and the current acceleration of the universe from a single $f(T)$ model and the combination of Friedmann's equations. In the quest for a simple equation with fewer unknowns after combining the Friedmann's equations, we will introduce the scalar field which, according to Noether's theorem, has an energy-momentum tensor suitable for this purpose. Recent works support the use of the scalar field in such investigations. For example, the author in [32] has provided a theoretical model of $f(R)$ gravity in which it is possible to describe, in a unified way, inflation, an early and a late dark energy era. The scalar field in these types of investigations is known as the axion. Finally, the quadratic model, in its algebraic form, will make it easy to obtain, by means of a single parameter, plausible results in the teleparallel theories in order to further support the idea of modification of gravity when comparing the results to observational data. In the present work, the comparison with observational data will be qualitative and is just used to prove that these data which sustain the inflationary scenario as well as the current expansion can be met for a suitable choice of the quadratic model parameter and other parameters generated from the differential equation resolution. Thus, statistical fitting or likelihood, which are very interesting tools to constrain theoretical parameters, will not be used in this investigation. We also emphasize here that the inflationary scenario and the dark energy dynamics will be described by means of the e-folding number Nand the redshift z, respectively.

The present paper is organized as follows: in Section 2, we introduce the f(T) theory by establishing the main equations. Section 3 is devoted to the description of the inflationary scenario from the main equations and to the comparison with observational data. Section 4 addresses dark energy and the cosmological scope of the obtained results. The paper ends with Section 5 which presents the conclusion.

MAIN EQUATIONS IN THE COUPLING MODIFIED TELEPARALLEL THEORY AND SCALAR FIELD

The modified teleparallel theory f(T) action is expressed as [33]

$$S=\frac{1}{4\kappa^2}\int d^4x h f(T)+\int d^4x h\mathcal{L}_M, \tag{1}$$

where $h = |\det(h^a{}_\mu)|$ is equivalent to $\sqrt{-g}$in General Relativity, $\kappa^2=\frac{16\pi G}{c^4}$, $\mathcal{L}_M$ is the Lagrangian of the matter field. Then, the variation of this action with respect to the tetrads $h^a{}_\mu$ gives

$$\frac{1}{h}\partial_\mu(hS_a{}^{\mu\nu})f_T(T)-h_a{}^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\mu\nu}f_T(T)+A^i{}_{a\mu}S_i{}^{\mu\nu}f_T(T)+S_a{}^{\mu\nu}\partial_\mu(T)f_{TT}(T)+\frac{1}{4}h_a{}^\nu f(T)=\frac{1}{4\kappa^2}T_a^\nu, \tag{2}$$

where $f_T(T) = df(T)/dT$, $f_{TT}(T) = d^2f(T)/dT^2$ and T_a^ν represents the energy-momentum tensor. In this study, we consider a universe described by the Friedmann-Lemaitre-Robertson-Walker metric, given by

$$ds^2=dt^2-a^2(t)\left(dx^2+dy^2+dz^2\right), \tag{3}$$

where a(t) denotes the scale factor. The scalar torsion related to the metric in Eq. (3) is given by

$$T=-6H^2(t), \tag{4}$$

where H(t) is the Hubble parameter. In the present work, we suppose that the universe is filled with a perfect fluid powered by the scalar field φ. In the context of the Friedmannn-Lemaitre-Robertson-Walker metric (31), the appropriate form of the energy momentum tensor of a perfect fluid is given by

$$T_{\mu\nu}= (\rho+ p)u_\mu u_\nu- pg_{\mu\nu} \tag{5}$$

where $g_{\mu\nu}$and u_ν, are the metric tensor and the 4-vector characterizing a co-mobile observer, respectively, and hand ρand p are the global energy density and the pressure of the universe content, respectively. Under these previous considerations, one can extract the Friedmannn- like equations of covariant modified Telleparallel theory

$$\kappa^2\rho = 6H^2f_T+\frac{1}{4}f \quad \text{and} \quad \kappa^2 p=48\dot{H}H^2f_{TT}-(2\dot{H}+6H^2)f_T-\frac{1}{4}f \tag{6}$$

In the present description, the only component of the universe content is supposed to be the scalar field. Indeed, the energy momentum tensor of the scalar field coming from the Noether's theorem is given by

$$\mathcal{T}_{\mu\nu}=\epsilon\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}\left[\frac{\epsilon}{2}\partial_\beta\phi\partial^\beta\phi-V(\phi)\right] \tag{7}$$

Here, V (φ) is the potential of the scalar field. By making use of the previous metric, we deduce from Eq. (7), the energy-density and the pressure of the scalar field, as in several works such as [34]-[37].

$$\rho=\frac{1}{2}\epsilon\dot{\phi}^2+V(\phi) \text{ and } p=\frac{1}{2}\epsilon\dot{\phi}^2-V(\phi) \tag{8}$$

So, the system of equations translating the interaction between the scalar field and the geometry in the framework of the modified theory is

$$\begin{aligned}\kappa^2\left(\frac{\epsilon}{2}\dot{\phi}^2+V(\phi)\right) &= 6H^2f_T+\frac{1}{4}f,\\ \kappa^2\left(\frac{\epsilon}{2}\dot{\phi}^2-V(\phi)\right) &= 48\dot{H}H^2f_{TT}-(2\dot{H}+6H^2)f_T-\frac{1}{4}f\end{aligned} \tag{9}$$

The conservation equation $\dot{\rho} + 3H(\rho + p) = 0$, in the present context, leads to the following equation called Klein-Gordon equation [33]

$$\epsilon\ddot{\phi} + 3H\epsilon\dot{\phi} + V'(\phi) = 0 \quad (12)$$

By adding the equations Eq. (9) and Eq. (11); we have

$$48\dot{H}H^2 f_{TT} - 2\dot{H}f_T = \kappa^2\epsilon\dot{\phi}^2 \quad (13)$$

We are dealing here with a cosmological investigation based on the following quadratic model [29].

$$f(T) = T + \lambda T^2 \quad (14)$$

Under the algebraic function Eq. (14), the motor equation Eq. (13) becomes

$$120\lambda\dot{H}H^2 = \kappa^2\epsilon\dot{\phi}^2 \quad (15)$$

INFLATIONARY SCENARIO FROM QUADRATIC F(T) MODEL

In an attempt to describe the inflationary scenario, we introduce the following operator relating the e-folding number Nto cosmic time t. Thus, according to this consideration, the scalar field and its potential, as well as the Hubble parameter, become functions of the e-folding number N. The transformation is introduced as [49].

$$\frac{d}{dt} = H(N)\frac{d}{dN} \quad (16)$$

The equation Eq. (15) can be rewritten as

$$120\lambda H'(N)H(N) = \kappa^2\epsilon(\phi'(N))^2 \quad (17)$$

Here, the prime (′) denotes the derivative with respect to the e-folding number. In the same way, we express the Klein-Gordon equation in Eq. (12) as

$$\epsilon H(N)^2\phi''(N) + \epsilon(H(N)H'(N) + 3H(N)^2)\phi'(N) + V'(\phi) = 0 \quad (18)$$

where $V'(\phi)$means the derivative of the potential with respect to the scalar field ϕ. The previous equation will be solved with the goal of expressing the inflationary observables. Let us remark that Eq. (17) is a differential equation of two separated e-folding number functions, $H(N)$and $\phi(N)$. The function $H(N)$is related to the space-time geometry, whereas $\phi(N)$is related to the matter. The knowledge of these two functions is crucial, or simply indispensable, in an attempt to have the other cosmological quantities such as the scalar field potential and the inflationary observables. But here, we have only one equation, Eq. (17), to do the job. The first idea can consist of choosing a cosmological ansatz expression for one of them and solving Eq. (17) to find the second. For example, according to the literature (see [50], [51]), it is possible to deduce the Hubble parameter $H(N)$describing the power-law expansion and the de Sitter expansion. Such an approach is carried out in our recent work [52]. Here, after analysing the form of Eq. (17), we think about a simple mathematical way to solve it without supposing the Hubble parameter. Through such an approach, it is also possible to provide another Hubble parameter expression in the inflationary scenario description. Our approach to get two unknown functions from only one differential equation consists in introducing a constant c.

$$120\lambda H'(N)H(N) = \kappa^2(\phi'(N))^2 = c \quad (19)$$

The equation Eq. (19) solutions are the functions H(N) and ϕ (N). In order to have real solutions, the positivity of the parameter c is required ($c > 0$). After solving the decoupled differential equations, one has

$$H(N) = \frac{\sqrt{cN + 120c_1\lambda}}{2\sqrt{15}\sqrt{\lambda}} \quad (20)$$

$$\phi(N) = \frac{\sqrt{c}N}{\sqrt{\epsilon}k} + c_2 \quad (21)$$

where c_1 and c_2 are integration constants. The relation in Eq. (21) permits to express the e-folding number as a function of the scalar field. By making use of Eq. (20) in the Klein-Gordon equation in Eq. (18), one can extract the potential of the scalar field as follows

$$V(\phi) = -\frac{c^{3/2}\phi - 6c\kappa c_2\phi + 3c\kappa\phi^2 + 720\sqrt{c}\lambda c_1\phi}{120\kappa\lambda} + c_3 \quad (22)$$

Here, we have posed $\epsilon = 1$ (quintessence-like evolution [28]) to avoid confusion with the slow-roll parameters.
The expressions for the slow-roll parameters with respect to the canonical scalar field potential $V(\phi)$ are given by [44]

$$\varepsilon \equiv \frac{1}{2\kappa^2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \quad , \quad \eta \equiv \frac{1}{\kappa^2}\frac{V''(\phi)}{V(\phi)} \quad , \quad \xi^2 \equiv \frac{1}{\kappa^4}\frac{V'(\phi)V'''(\phi)}{(V(\phi))^2} \quad (23)$$

For the scalar field models, the spectral index n_s, of curvature perturbations, the tensor-to-scalar ratio r of the density perturbations and the running of spectral index α_s are expressed as [45]

$$n_s - 1 \sim -6\varepsilon + 2\eta, \quad r = 16\varepsilon, \quad \alpha_s \equiv \frac{dn_s}{dln\kappa} \sim 16\varepsilon\eta - 24\varepsilon^2 - 2\xi^2 \quad (24)$$

Depending on the scalar field potential, which is a function of the scalar field, the previous observables can be expressed in terms of the scalar field. But from the fact that the scalar field is directly related to the e-folding number through Eq. (21), these observables can be expressed as e-folding number functions. Such an approach makes possible the fitting of these observables with observational data. We firstly express the slow-roll parameters.

$$\varepsilon = \frac{\left(c^{3/2} + 6ck(\phi - c_2) + 720\sqrt{c}\lambda c_1\right)^2}{2k^2\left(\sqrt{c}\phi\left(3\sqrt{ck}(\phi - 2c_2) + c + 720\lambda c_1\right) - 120k\lambda c_3\right)^2} \quad (25)$$

$$\eta = -\frac{6c}{k\left(120k\lambda c_3 - \sqrt{c}\phi\left(3\sqrt{c}k(\phi - 2\sigma) + c + 720\lambda s\right)\right)} \quad (26)$$

$$\xi^2 = 0 \quad (27)$$

The observables are obtained as function of the e-folding number as follows

$$r = \frac{8c(6cN + c + 720\lambda c_1)^2}{\left(3c^2N^2 + c^2N - 3c\kappa^2c_2^2 + 720c\lambda Nc_1 + \sqrt{c}\kappa c_2(c + 720\lambda c_1) - 120\kappa^2\lambda c_3\right)^2} \quad (28)$$

$$\eta = -\frac{3c(6cN + c + 720\lambda c_1)^2}{\left(3c^2N^2 + c^2N - 3c\kappa^2c_2^2 + 720c\lambda Nc_1 + \sqrt{c}\kappa c_2(c + 720\lambda c_1) - 120\kappa^2\lambda c_3\right)^2} + \frac{12c}{3c^2N^2 + c^2N - 3c\kappa^2c_2^2 + 720c\lambda Nc_1 + \sqrt{c}\kappa c_2(c + 720\lambda c_1) - 120\kappa^2\lambda c_3} + 1 \quad (29)$$

$$\alpha_s = \frac{A(N)}{B(N)} \quad (30)(31)$$

With

$$A(N) = 6c^2(6cN + c + 720\lambda c_1)^2\Big[8\Big(3c^2N^2 + c^2N - 3c\kappa^2c_2^2 + 720c\lambda Nc_1 + \sqrt{c}\kappa c_2(c + 720\lambda c_1) - 120\kappa^2\lambda c_3\Big) - (6cN + c + 720\lambda c_1)^2\Big] \quad (32)$$

$$B(N) = \left(3c^2N^2 + c^2N - 3c\kappa^2c_2^2 + 720c\lambda Nc_1 + \sqrt{c}\kappa c_2(c + 720\lambda c_1) - 120\kappa^2\lambda c_3\right)^4 \quad (33)$$

Several observational data are investigated on these parameters. The use of these data remains a viable way to constrain theoretical models. We present here the recent observations on the spectral index n_s, the tensor-to-scalar ratio r, and the running of the spectral index α_s. The recent data of the Planck satellite [4] suggested $n_s = 0.9603 \pm 0.0073$(68%CL), $r < 0.11$(95%CL), and $\alpha_s = -0.0134 \pm 0.0090$(68%CL) [Planck and WMAP [41]; [40]], whose negative sign is at 1.5σ. The BICEP2 experiment [4] implies $r = 0.20^{+0.07}_{-0.05}$(68%CL). It is mentioned that discussions exist on how to subtract the foreground, for example in [4],[39]. Recently, progress has also appeared in [42] to ensure the BICEP2 declarations. It has been also remarked that the representation of α_s is given in [43].

The contribution of the quadratic model in the theoretical description of these observables is clearly shown through figures 1 to 3. Indeed, figure 1 reveals that through the quadratic model, the tensor-to-scalar ratio r typically decreases with the increase of the e-folding number N. Under this evolution, several observational data on the tensor-to-scalar ratio r can be met. This means that as inflation progresses, the contribution of gravitational waves, carried by the tensor-to-scalar ratio r, becomes less important compared to scalar perturbations. For example, the Planck Collaboration's 2018 results in [46] and the BICEP2 experiment [4] suggestions support our theoretical prediction under the quadratic model and near the end of inflation namely N = 60. The graph on the right of figure 1 illustrates the fact that the teleparallel predictions are very far from those of the observational experiments. Such results strengthen the idea of the modification of the teleparallel theory of gravity. In figure 2, the spectral index promoted by the quadratic model increases and near N = 60, it leads to observational data established in [46]. At the same time, the result given by the pure teleparallel theory (the right graph of figure 2) on the spectral index does not satisfy any observation data. Finally, figure 2 shows that the running of spectral index predicted by the quadratic model, especially near N = 60, are consistent with the Planck data mentioned [43] and [46] whereas the evolution of this observable is chaotic in the case of pure teleparallel.

To show further the consistency of the curves plotted in these figures, we extract from them some values of the observables that can be verified from appropriately referenced works. Table 1 presents these values, which are localizable in a set of values presented in the cited references. Another argument which supports the consistency of the present theoretical description comes from [47]. After providing the observational data on these inflationary observables, they also conclude that in the framework of single-field inflationary models with Einstein gravity, the slow-roll models with a concave potential $V'(\phi) < 0$ are increasingly favoured by data. We note here that the observables described in Figures 1-3 and Table 1 are expressed from the polynomial potential of Eq. (22). By the way, one has $V'(\phi) = -\frac{c}{20\lambda}$.

We have justified in the paragraphs above that, for reasons of consistency in Eq. (19) resulting from the Friedmann equations, the parameter cmust satisfy the condition $c > 0$. The resolution of this equation leads to the Hubble parameter presented in Eq. (20). As the rate of universe expansion, it is generally a measurable real quantity. So, from the expression in Eq. (20), the positivity of the parameter λis also required ($\lambda > 0$). It therefore follows that $V'(\phi) < 0$. In conclusion, the scalar field potential decreases during inflation, leading to the possibility of having a minimum potential. According to the general review of inflationary cosmology investigated in [44], the step where the scalar field falls into its minimum potential marks the beginning of elementary particle creation and the reheating process. At this level, the energy of the scalar field is transformed into heat, allowing the universe to rejoin the standard hot Big Bang model. The explanation of this stage is not concerned with this work, although its mention marks the end of inflation, which is crucial in a description unifying inflation and late-time acceleration. In the present subject of unification, the same condition is also required in [52], where ultrarelativistic matter is used to destabilize inflation and eventually leads to the exit from the inflationary stage.

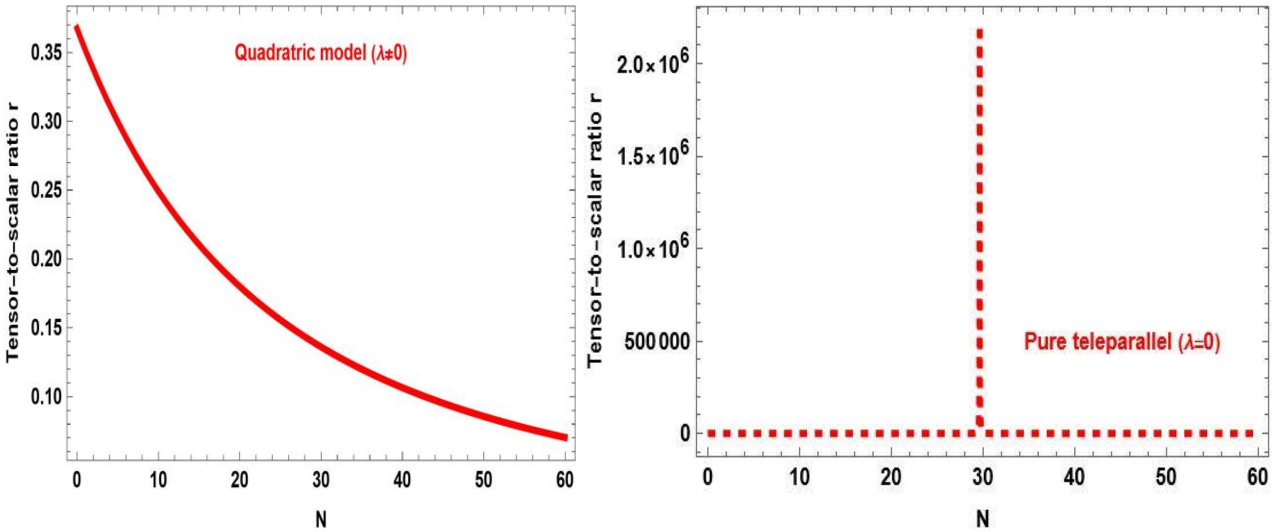

Figure 1: Evolution versus the e-folding Number N of the tensor-to-scalar ratio r for the quadratic model ($\lambda = 10$ leading to the left panel) and pure teleparallel theory ($\lambda = 0$ leading to the right panel). The curves are obtained for $c = 0.01$, $c_1 = -2$, $c_2 = 3$, $c_3 = 2$ and $\kappa = 1$.

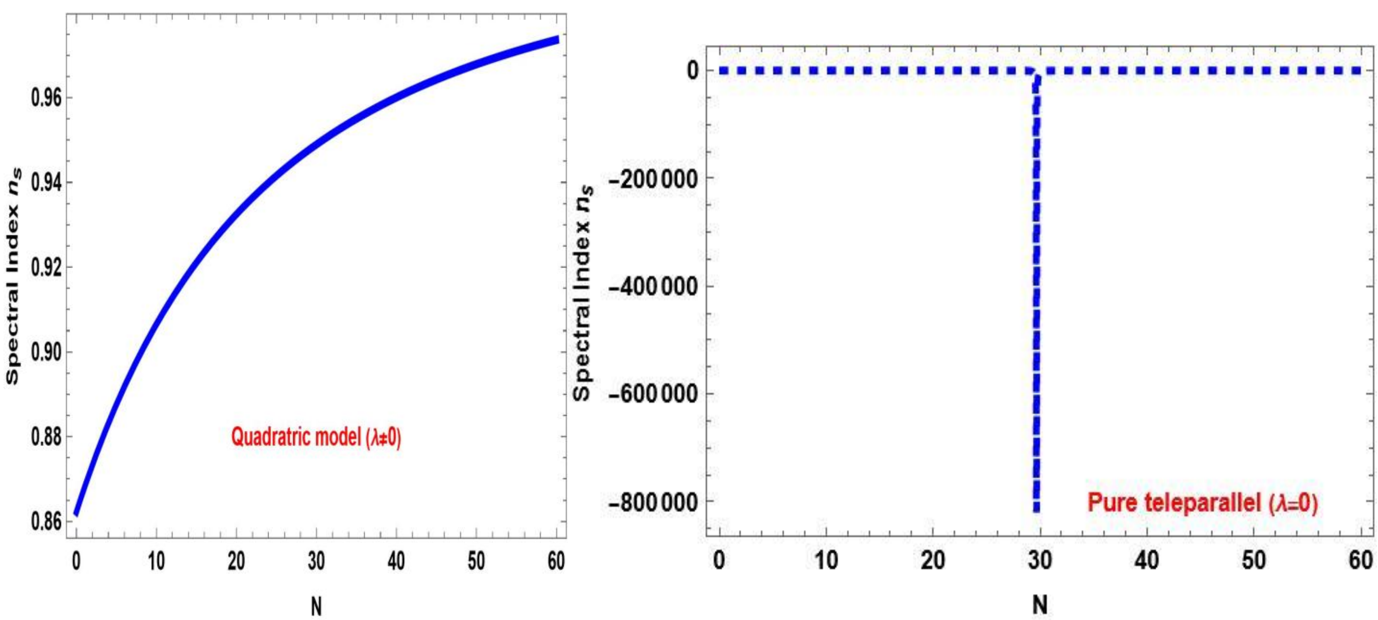

Figure 2: Evolution versus the e-folding Number N of the spectral index n_s for the quadratic model ($\lambda = 10$ leading to the left panel) and pure teleparallel theory ($\lambda = 0$ leading to the right panel). The curves are obtained for $c = 0.01$, $c_1 = -2$, $c_2 = 3$, $c_3 = 2$ and $\kappa = 1$.

N	Tensor-to-scalar ratio r	Spectral index n_s	Running of spectral index α_s	Refs.
40	0.1064	0.9601	-0.0011	[4,44]
45	0.0952	0.9643	-0.0008	[47,46]
50	0.0856	0.9679	-0.0007	[47,46]
55	0.0774	0.9709	-0.0006	[47,46]
60	0.0702	0.9736	-0.0005	[47,46]

Table 1: Observable values from the quadratic teleparallel model. These values are deduced from the figures Fig.1 toFig.3 for $c = 0.01$, $c_1 = -2$, $c_2 = 3$, $c_3 = 2$ $\kappa = 1$ and $\lambda = 10$

DARK ENERGY DESCRIPTION FROM QUADRATIC TELEPARALLEL MODEL

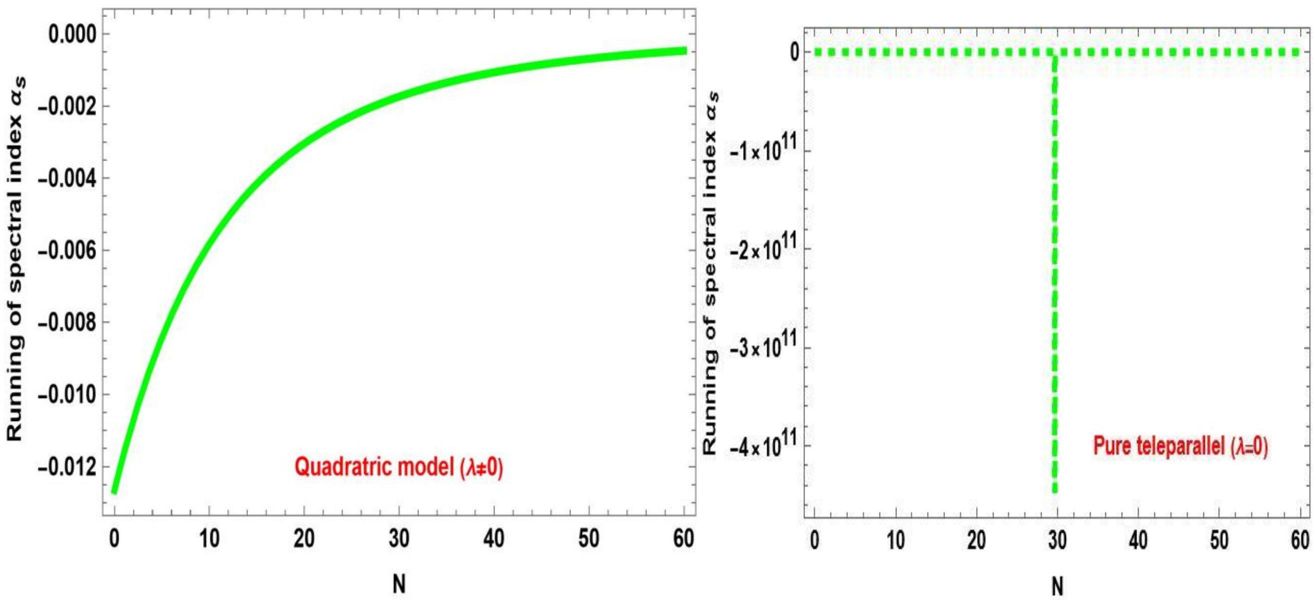

Figure 3: Evolution versus the e-folding Number N of the running of spectral index α_s for the quadratic model ($\lambda = 10$ leading to the left panel) and for teleparallel theory ($\lambda = 0$ leading to the right panel). The curves are obtained for $c = 0.01$, $c_1 = -2$, $c_2 = 3$, $c_3 = 2$ and $\kappa = 1$.

The dark energy description is one of the most currently attractive subjects in cosmology. Like several investigations, the introduction of the scalar field gives a real way to deal with the topic in the context of a modified theory of gravity. In the present section, the Friedmann equations powered by the quadratic model with the scalar field as the only universe content will be solved to provide the energy density, the pressure, and the equation-of-state parameter. We recall that the sum of the two Friedmann equations has generated Eq. (15) when the quadratic $f(T)$ model is applied. In the same approach as the previous section, this equation is solved and gives.

$$H(t) = \frac{\sqrt[3]{ct + 120\lambda s}}{2\sqrt[3]{5}\sqrt[3]{\lambda}} \qquad (34)$$

$$\phi(t) = \frac{\sqrt{c}t}{\kappa\sqrt{\epsilon}} + \sigma \tag{35}$$

Here, c, s and σare integration constants. The resolution of the Klein-Gordon equation (12) gives

$$V(\phi) = -\frac{9\left(\sqrt{c}\kappa\sqrt{\epsilon}(\phi-\sigma)+120\lambda s\right)^{4/3}}{8\sqrt[3]{5}\kappa^2\sqrt[3]{\lambda}} + v \tag{36}$$

with v, the integration constant. We can now provide the energy density and the pressure of the scalar field as functions of cosmic time t.

$$\rho(t) = \frac{c}{2\kappa^2} - \frac{9(ct+120\lambda s)^{4/3}}{8\sqrt[3]{5}\kappa^2\sqrt[3]{\lambda}} + v, \qquad p(t) = \frac{c}{2\kappa^2} + \frac{9(ct+120\lambda s)^{4/3}}{8\sqrt[3]{5}\kappa^2\sqrt[3]{\lambda}} - v \tag{37}$$

To made description based on the recent data on the cosmological parameters, we aim provide these previous quantities in term of redshift z. In connection with redshift, the Hubble parameter is defined by [28].

$$H(t) = -\frac{1}{1+z}\frac{dz}{dt} \tag{38}$$

By using Eq. (34) and Eq. (38) one obtains the cosmic time t as a function of the redshift z.

$$t = 4\left(\frac{\sqrt[4]{10}\sqrt[4]{b^3\lambda - 3b^2\lambda\log(z+1) + 3b\lambda\log^2(z+1) - \lambda\log^3(z+1)}}{3^{3/4}\sqrt[4]{c}} + \frac{60\lambda}{c}\right) \tag{39}$$

Here b is also an integration constant. By the way, one has

$$H(z) = \frac{\sqrt[3]{\sqrt[4]{30}c^{3/4}\sqrt[4]{\lambda(b-\log(z+1))^3} + 90\lambda(s+2)}}{\sqrt[3]{30}\sqrt[3]{\lambda}} \tag{40}$$

$$\rho(z) = \frac{1}{10}\left(-\frac{30^{2/3}\left(\sqrt[4]{30}c^{3/4}\sqrt[4]{\lambda(b-\log(z+1))^3} + 90\lambda(s+2)\right)^{4/3}}{\kappa^2\sqrt[3]{\lambda}} + \frac{5c}{\kappa^2} + 10v\right) \tag{41}$$

$$p(z) = \frac{1}{10}\left(\frac{30^{2/3}\left(\sqrt[4]{30}c^{3/4}\sqrt[4]{\lambda(b-\log(z+1))^3} + 90\lambda(s+2)\right)^{4/3}}{\kappa^2\sqrt[3]{\lambda}} + \frac{5c}{\kappa^2} - 10v\right) \tag{42}$$

From several works like [28,46,47], it is possible to know the current (z = 0) observational value of the Hubble parameter. For example, it is estimated as $H(0) = H_0 = 70.4 \pm 1.6$ km.s^{-1}. We use these values as initial conditions to extract the expression of the parameter b. Such an approach helps to reduce the number of parameters and to base our theoretical description on observational data. One has

$$b = \frac{30\lambda\left(-H_0^3+3s+6\right)^{4/3}}{c} \tag{44}$$

$$H(z) = \frac{\sqrt[3]{\sqrt[4]{30}\,c^{3/4}\sqrt[4]{\lambda\left(\frac{30\lambda\left(-H_0^3+3s+6\right)^{4/3}}{c}-\log(z+1)\right)^3}+90\lambda(s+2)}}{\sqrt[3]{30}\,\sqrt[3]{\lambda}} \tag{45}$$

$$\rho(z) = -\frac{30^{2/3}\left(\sqrt[4]{30}\,c^{3/4}\sqrt[4]{\lambda\left(\frac{30\lambda\left(-H_0^3+3s+6\right)^{4/3}}{c}-\log(z+1)\right)^3}+90\lambda(s+2)\right)^{4/3}}{10\kappa^2\sqrt[3]{\lambda}} + \frac{c}{2\kappa^2}+v \tag{46}$$

$$p(z) = \frac{30^{2/3}\left(\sqrt[4]{30}\,c^{3/4}\sqrt[4]{\lambda\left(\frac{30\lambda\left(-H_0^3+3s+6\right)^{4/3}}{c}-\log(z+1)\right)^3}+90\lambda(s+2)\right)^{4/3}}{10\kappa^2\sqrt[3]{\lambda}} + \frac{c}{2\kappa^2}-v \tag{47}$$

We now plot these quantities versus the redshift z. We provide the evolution of the Hubble parameter in order to test its consistency with observational data and to compare its behaviors to those already investigated in the literature. Moreover, after the energy density and the pressure depicted in figure 5, the equation of state parameters $\omega(z) = p(z)/\rho(z)$ is also provided in figure 6.

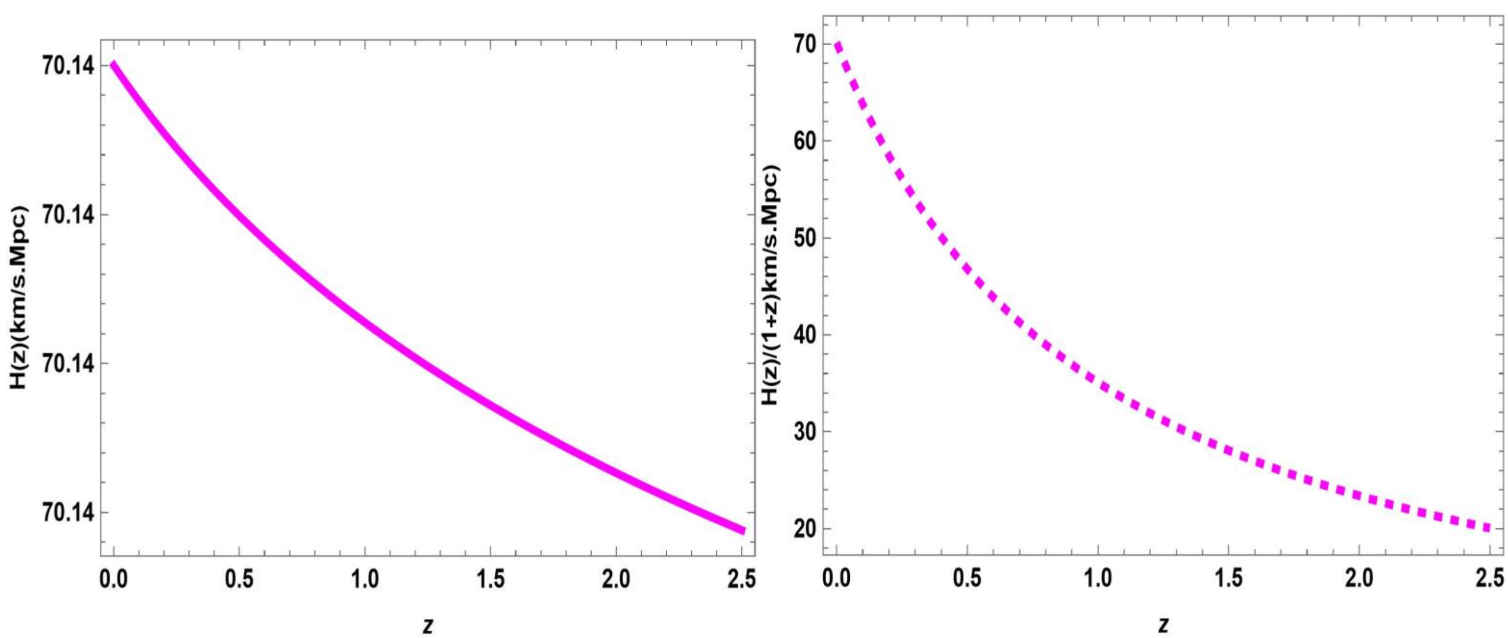

Figure 4: Evolution versus the redshift z of the Hubble parameter. The curve is obtained for $H_0 = 70.14$; $s = 220000$; $\lambda = -20$; $c = 10$.

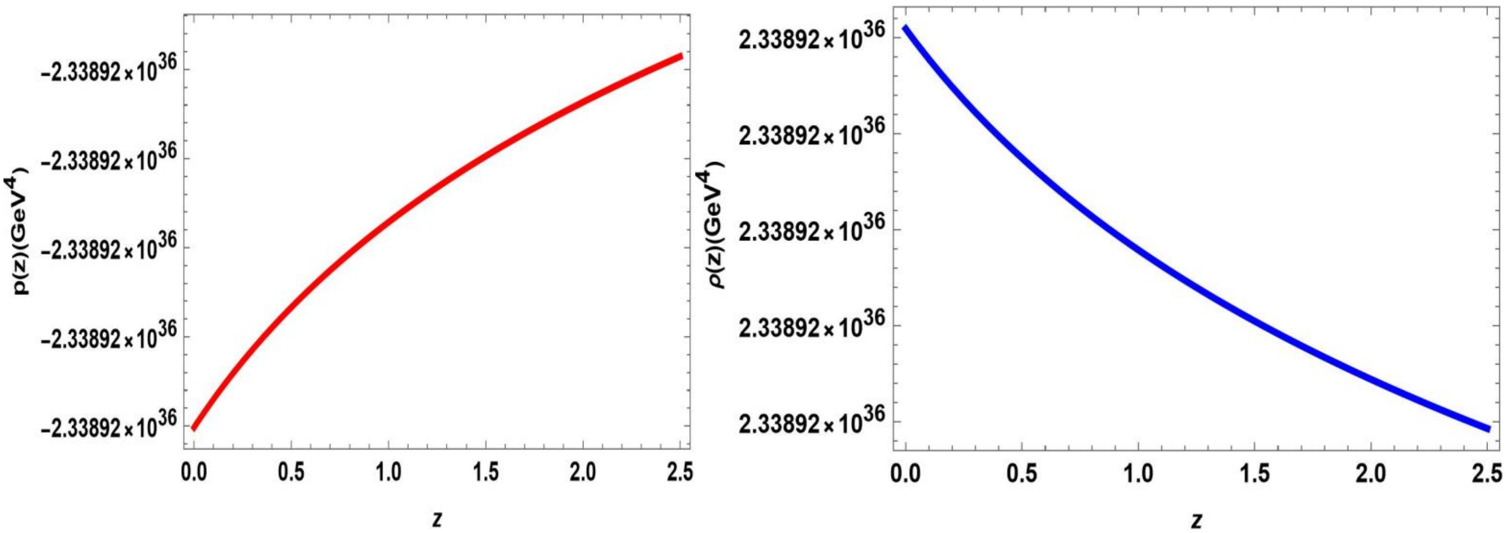

Figure5: Evolution versus the redshift z of the energy density and the pressure of the scalar field in quadratic teleparallel model constrained by observational data. The curves are obtained for H_0= 7014; s = 220000; λ = -20; c = 10; υ = 100; k = $\sqrt{\frac{1.8626}{10^{26}}}$SI

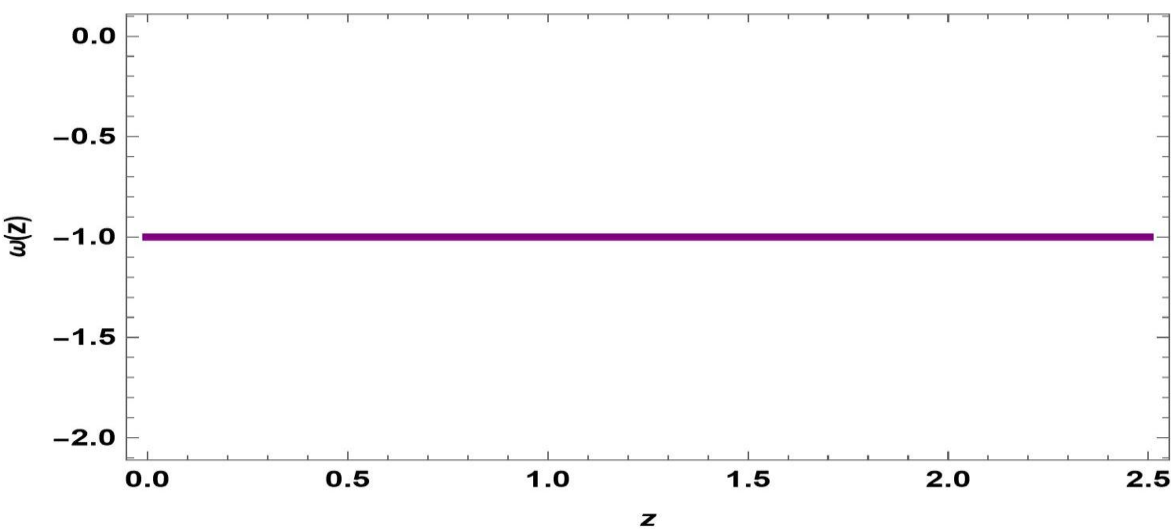

Figure 6: Evolution versus the redshift z of the equation of state parameter of the scalar field ω_ϕ. The curve is obtained for H_0 = 7014; s = 220000; λ = -20; c = 10; υ = 100; k = $\sqrt{\frac{1.8626}{10^{26}}}$SI

COSMOLOGICAL SCOPE

One of the main goals of theoretical investigation is to defend observational predictions. Here, the Hubble parameter reconstructed in Eq. (35) is constrained to give the present observational value of this cosmological parameter. The result is presented in the figure 4 where under the choice $H_0 = H(0) = 70.14$ km.S^{-1}.Mpc^{-1} (see [28]), the Hubble parameter H(z) decreases slightly with the increase of the redshift z. This decreasing with the redshift aligns with the results obtained in interesting studies in [28, 46, 47]. Although this decrease, the variation of the Hubble parameter is not significant. This means that our model promotes a constant Hubble parameter whose value is given by observational data. Furthermore, the curve of H(z)/(1+z) in figure 4 presents a concavity facing upwards like several works [28]. This analysis on the Hubble parameter has permitted us to know the conditions under which, our model can lead to the present observational value of the Hubble parameter. So, the free parameters of our model are constrained in order to deal with one of the great cosmological and astrophysical problems: the problem of dark energy.

In the present investigation, it is important to recall that the considered candidate of the dark energy is the scalar field. It represents an alternative object to the cosmological constant which the most plausible candidate of dark energy. Recent works report that, in dynamical dark energy models, the equation of state of the dark energy changes over time [7, 8]. These models include but are not limited to quintessence, k-essence, and phantom-type scalar field models, where generally a scalar field is coupled with matter minimally or non-minimally, with an associated potential which can generate sufficient negative pressure to drive the accelerated expansion of the universe. It is suggested in current observations that the equation of state of the scalar field might have a phantom barrier ($\omega_\phi = -1$) crossing in the recent past [10, 11].In the present work, and under the value of the free parameters for which the Hubble parameter gives its current observational value, we depict versus the redshift, the pressure, the energy density and the equation-of-state parameter of the scalar field. The evolution of these quantities is presented in Figures 5-6. The scalar field pressure is negative and increases with redshift, whereas the energy density decreases with redshift. Although these variations, their values with respect to the redshift are very near their present value ($z = 0$). The equation of state of the scalar field is therefore practically constant and gives ($\omega_\phi = -1$). In conclusion, the scalar field behaves like the constant cosmological constant Λ in the framework of quadratic teleparallel model.

To strengthen all these results, it will be interesting to follow the evolution of the scalar potential and compare it to existing results in the context of the accelerated expansion of the universe. Firstly, we express as redshift function, the scalar field potential in Eq. (36) and we obtain

$$V(z) = \frac{1}{10}\left(10v - \frac{30^{2/3}\left(\sqrt[4]{30}c^{3/4}\sqrt[4]{\lambda\left(\frac{30\lambda\left(-H_0^3+3s+6\right)^{4/3}}{c} - \log(z+1)\right)^3} + 90\lambda(s+2)\right)^{4/3}}{\kappa^2\sqrt[3]{\lambda}}\right) \quad (48)$$

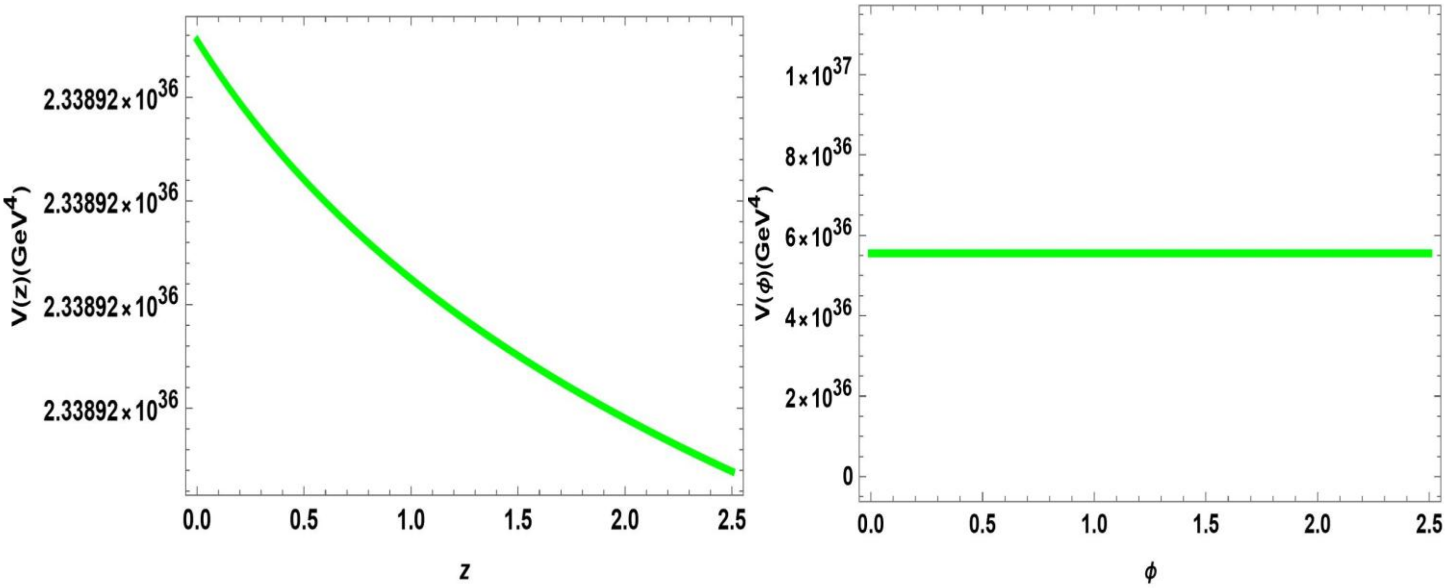

Figure 7: Evolution versus the redshift z (left panel) and versus the scalar field ϕ (right panel) of the scalar field potential. The graphs are obtained for $H_0 = 70.14$; s = 220000; $\lambda = -20$; c = 10; v = 100; $k = \sqrt{\frac{1.8626}{10^{26}}}$SI

Recent works [8, 48] indicate under dark energy domination, the potential must remain very flat and vary extremely slowly with z, except dynamic quintessence models where a gentle decay is possible. When constraining the quadratic teleparallel model in order to give results aligned with current observational data, it makes the quintessence scalar field giving results near the constant cosmology. First, we obtain from the left panel of figure Fig.7 where the scalar field potential decreases with the redshift. Moreover, this variation is very low and is clearly shown through the right panel of figure Fig.7 when depicting this potential versus the scalar field. So, the scalar field potential behaviours meet some literature prediction in the description of dark energy domination era.

Conclusion :-

The present investigation is devoted to the cosmological implications of the quadratic teleparallel theory, with goal of providing a unified description of inflation and the dark-energy-dominated era. The universe is assumed to be filled with a scalar field, and a metric satisfying the cosmological principle is adopted. All results are obtained from the resolution of the Friedmann equations induced by quadratic model. The so-called resolution leads to explicit expressions for the Hubble parameter and the scalar field. By taking into consideration the Klein-Gordon equation, we extract the scalar field potential corresponding to each dynamical era investigated in this work.

In inflationary scenario description, the slow-roll parameters and inflationary observables are derived as e.folds number function. The comparison with observational data is made for suitable choice of the free parameters (quadratic model parameter and the integration constants). Indeed, for $\lambda = 20$, the spectral index $n_s$approaches 0.96 near $N = 60$, which agrees with Planck and WMAP observations. In the same time, the tensor-to-scalar ratio ris consistent with BICEP2 results. We also numerically show that the expected values of these observables cannot be reached within the pure teleparallel model ($\lambda = 0$) due to their chaotic behaviour in this context.

When describing the late time expansion, the analysis is performed by mean of the redshift z. The Hubble parameter, the energy density, the pressure, and scalar field potential are reconstructed and constrained to reproduce the current observational value of the Hubble parameter. The model predicts an approximately constant equation-of-state parameter so reproducing the ΛCDM model feature. This indicates that the scalar field behaves effectively as dark energy candidate. The scalar field potential varies slowly with redshift and remains nearly constant, which is in perfect agreement with expected results in a dark-energy-dominated universe.

Overall, the quadratic $f(T)$ model allows the scalar field to behave both as an inflaton and as a dark energy candidate close to the cosmological constant Λ. Although the scalar field mimics some properties of an axion, it cannot be identified as such, since it evolves dynamically during inflation, unlike the axion field. Finally, the present work does not address the transition between inflation and the radiation era. Future investigations will focus on the conditions required for a stable transition from inflation to radiation domination and to the current accelerated expansion.

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