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RESEARCH ARTICLE

SOME STATISTICAL ANALYSIS OF FINITE FOURIER TRANSFORM AND PERIODOGRAM WITH DISCRETE TIME SERIES

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Abstract

This paper defines the Finite Fourier Transform and the periodogram, and investigates several of their statistical properties. Key definitions, theorems, and results are presented to establish their asymptotic behavior in the context of discrete-time series analysis.

Introduction:-

Fourier analysis has been widely recognized as a powerful tool in time series studies. Multiple researchers, including Brillinger [1] and Elhassancin [6], have contributed to its theoretical development. Extending this line of inquiry, Ghazal et al. [12] examined the asymptotic properties of discrete stable time series with missing observations between two vector-valued stochastic processes. Ghazal et al. [13] further studied the statistical properties of the periodogram for two vector-valued stability series with missed observations. El-Deoskey et al. [8] investigated the extended finite Fourier transform for firmly fixed continuous time series with missing data, while El-Deoskey et al. [7] analyzed linearly immutable continuous time series modeled as bivariate stochastic processes with vector values, focusing on their distinguishing features. Additionally, Alargt and Ben Aros [14] explored the asymptotic properties of the finite Fourier transform for filtered series. Building on these works, the present paper conducts a statistical analysis of the finite Fourier transform and periodogram within the framework of discrete-time series.

Suppose we observe a stretch of T consecutive values $X(t), t=0,...,T-1$ from an r-vector-valued strictly stationary series with mean function $EX = C_X$ and spectral density matrix

$$f_{XX}(\lambda)=\sum_{u=-\infty}^{\infty} C_{XX}(u)\exp(-i\lambda u), \quad -\infty<\lambda<\infty .$$

This paper studies several statistical properties of the finite Fourier transform and the periodogram derived from such series.

Definitions:-

Definition 1(Brillinger[1])

Given a sequence X(t) for t=0,1,…,T−1, the finite Fourier transform of this segment is defined as

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$$d_X^{(T)}(\lambda) = \sum_{t=0}^{T-1} X(t) \exp(-it\lambda), \; -\infty < \lambda < \infty. \quad (1)$$

Definition 2(Brillinger[1])

A standard estimator for the power spectrum $f_{XX}(\lambda)$ is the periodogram, given by

Estimating the power spectrum $f_{XX}(\lambda)$, is suggested by the periodogram that is defined by

A common estimator for the power spectrum $f_{XX}(\lambda)$ is the periodogram, which is calculated as:

$$I_{XX}^{(T)}(\lambda) = (2\pi T)^{-1} \left| d_X^{(T)}(\lambda) \right|^2 \quad (2)$$

typically considered for frequencies $\lambda \neq 0, \pm\pi, \pm 2\pi, ...$

Definition 3(Brillinger[1])

A data window function $h_a^{(T)}(t)$ is used to taper the series. It is defined as $h_a^{(T)}(t) = h_a^{(T)}(\frac{t}{T})$ for $t \in (0, T)$, is bounded and of bounded variation, and vanishes outside the interval [0, T].

Theorems and Proofs:-

Theorem1

Assume $X(t)$ is a strictly stationary discrete-time series, and let $d_X^{(T)}(\lambda)$ be its finite Fourier transform as defined in (1). When a data window $h_a^{(T)}(t)$ is applied, the asymptotic distribution of the transform is as follows:

1) $d_X^{(T)}(\lambda) \sim \mathrm{N_r}\left(TH_a(0)C_a, 2\pi TH_{ab}(0) f_{ab}(\lambda)\right)$. if $\lambda = 0, \pm 2\pi, ...$

2) $d_X^{(T)}(\lambda) \sim \mathrm{N_r}\left(0, 2\pi TH_{ab}(0) f_{ab}(\lambda)\right)$. if $\lambda = \pi, \pm 3\pi, ...$

Where $TH_a(0) \approx \sum_t h_a(\frac{t}{T}) = H_a^{(T)}(0)$ and $TH_{ab}(0) \approx \sum_t h_a(\frac{t}{T}) h_b(\frac{t}{T}) = H_{ab}^{(T)}(0)$

Proof:|

Let $X(t)$ be a strictly stationary discrete-time series, and consider its windowed finite Fourier transform:

$$d_X^{(T)}(\lambda) = \sum_t^{T-1} h_a(\frac{t}{T}) X_a(t) \exp(-i\lambda t)$$

Calculation of the Mean:

Taking the expectation:

$$E\left[d_X^{(T)}(\lambda)\right] = \sum_t^{T-1} h_a(\frac{t}{T}) e^{-i\lambda t} E\left[X_a(t)\right]$$

Since the process is stationary with mean C_a, this becomes:

$$E\left[d_X^{(T)}(\lambda)\right] = C_a \sum_t^{T-1} h_a(\frac{t}{T}) e^{-i\lambda t}$$

Let $H_a^T(0) = \sum_t^{T-1} h_a(\frac{t}{T})$.

When $\lambda = 0 \pm 2\pi, ...$, we have $\exp(-i\lambda t)$=1, yielding:

$$E\left[d_X^{(T)}(\lambda)\right] = C_a H_a^T(0) \approx TC_a H_a(0)$$

When $\lambda = \pi, \pm 3\pi, ...$, the term $\exp(-i\lambda t)$ alternates in sign, and for large T with a smooth taper, the sum tends to zero:

$$E\left(d_X^{(T)}(\lambda)\right) \to 0.$$

Calculation of the Covariance:
We compute:

$$C0v\left[d_a^{(T)}(\lambda), d_b^{(T)}(\lambda)\right] = \sum h_a(t_1/T) h_b(t_2/T) e^{-i\lambda(t_1-t_2)} C_{ab}(t_1 - t_2)$$

where $C_{ab}(u)$ is the cross-covariance function.

Let $u = t_1 - t_2$ and $t = t_2$. For sufficiently large T, we approximate:

$$\sum e^{-i\lambda(t_1-t_2)} C_{ab}(u) \approx 2\pi f_{ab}(\lambda),$$

and

$$\sum h_a((t+u)/T) h_b(t/T) \approx T \int h_a(x) h_b(x)\, dx = T H_{ab}(0).$$

Thus:

$$C0v\left(d_a^{(T)}(\lambda), d_b^{(T)}(\lambda)\right) = 2\pi T H_{ab}(0) f_{ab}(\lambda).$$

Asymptotic Distribution:
Under the given conditions, and invoking a central limit theorem for weighted sums of stationary sequences, $d_X^{(T)}(\lambda)$ converges in distribution to a multivariate normal:

Case 1: $\lambda = 0 \pm 2\pi, \ldots$

$$d_X^{(T)}(\lambda) \sim \mathrm{N_r}\left(TH_a(0)C_a, 2\pi TH_{ab}(0) f_{ab}(\lambda)\right).$$

Case 2: $\lambda = \pi, \pm 3\pi, \ldots$

$$d_X^{(T)}(\lambda) \sim \mathrm{N_r}\left(0, 2\pi TH_{ab}(0) f_{ab}(\lambda)\right).$$

This completes the proof.

Theorem2:-
Let $X(t), t = 0, \pm 1, \ldots$, be an r-vector-valued strictly stationary series with mean C_X and cross-covariance function $C_{XX}(u)$. Assume

$$\sum_u \left|C_{XX}(u)\right| < \infty$$

Let $I_{XX}^{(T)}(\lambda)$ be the periodogram defined in (2), and let $h_a^{(T)}(t)$ be a data window. Then:

$$EI_{ab}^{(T)}(\lambda) = \left[\int_{-\pi}^{\pi} H_a^{(T)}(\alpha) H_b(-\alpha) d\alpha\right]^{-1} \left[\int_{-\pi}^{\pi} H_a^{(T)}(\alpha) H_b(-\alpha) f_{ab}(\lambda - \alpha) d\alpha + H_a^{(T)}(\lambda) H_b^{(T)}(-\lambda) C_a C_b\right],$$

$$for -\infty < \lambda < \infty,\ a, b = 1, \ldots, r.$$

Proof:
Starting from the definition of the periodogram:

$$I_{ab}^{(T)}(\lambda) = \frac{1}{2\pi T} d_a^{(T)}(\lambda) \overline{d_b^{(T)}(\lambda)},$$

where $d_a^{(T)}(\lambda)$

is the windowed finite Fourier transform. Taking expectations:

$$E\left(I_{ab}^{(T)}(\lambda)\right) = \frac{1}{2\pi T} E\left(d_a^{(T)}(\lambda) \overline{d_b^{(T)}(\lambda)}\right).$$

Expanding the expectation:

$$E\left(d_a^{(T)}(\lambda)\overline{d_b^{(T)}(\lambda)}\right) = \sum h_a^{(T)}(t_1)h_b^{(T)}(t_2)e^{-i\lambda(t_1-t_2)}E\left(X_a(t_1)X_b(t_2)\right).$$

Since $E\left(X_a(t_1)X_b(t_2)\right)C_{ab}(t_1-t_2)+C_aC_b$, we separate the sum into two parts:

$$= \sum_{t_1,t_2} h_a^{(T)}(t_1)h_b^{(T)}(t_2)e^{-i\lambda(t_1-t_2)}C_{ab}(t_1-t_2) + \left(\sum_{t_1} h_a^{(T)}(t_1)e^{-i\lambda t_1}C_a\right)\left(\sum_{t_2} h_b^{(T)}(t_2)e^{i\lambda t_2}C_b\right).$$

The second term simplifies to $C_aC_bH_a^{(T)}(\lambda)H_b^{(T)}(-\lambda)$.

For the first term, let $u = t_1 - t_2$ **and** $t = t_2$ **:**

$$\sum_{t,u} h_a^{(T)}(t+u)h_b^{(T)}(t)e^{-i\lambda u}C_{ab}(u).$$

For large T, the inner sum over u approximates the spectral density:

$$\sum_u e^{-i\lambda u}C_{ab}(u) \approx 2\lambda f_{ab}(\lambda).$$

The outer sum over t approximates:

$$\sum_t h_a^{(T)}(t+u)h_b^{(T)}(t) \approx T\int h_a(x)h_b(x)e^{i\alpha u}dx,$$

which, after applying Parseval's identity, leads to:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H_a^{(T)}(\alpha)H_b(-\alpha)e^{i\alpha u}d\alpha .$$

Combining these approximations and integrating over α, we obtain:

$$EI_{ab}^{(T)}(\lambda) = \left[\int_{-\pi}^{\pi} H_a^{(T)}(\alpha)H_b(-\alpha)d\alpha\right]^{-1}\left[\int_{-\pi}^{\pi} H_a^{(T)}(\alpha)H_b(-\alpha)f_{ab}(\lambda-\alpha)d\alpha + H_a^{(T)}(\lambda)H_b^{(T)}(-\lambda)C_aC_b\right].$$

This completes the proof:-

Corollary (Asymptotic Unbiasedness of the Periodogram):

Under the assumptions stated in Theorem 1, and provided that

$\int h_a(u)h_b(u)du \neq o$ for $a,b = 1,\ldots,r$,

the expected value of the periodogram converges asymptotically as follows:

$$\underset{T\to\infty}{Lim}\, E\left[I_{XX}^{(T)}(\lambda)\right] = f_{XX}(\lambda),$$

provided that either:

1. $\lambda \neq 0(\mathrm{mod}2\pi)$ #0(mod2 π), or
2. the mean vector $C_X = 0$.

Interpretation:

- The periodogram serves as an asymptotically unbiased estimator of the spectral density matrix $f_{XX}(\lambda)$ under the stated conditions.

- If the process has a non-zero mean ($C_X \neq 0$) and λ is near frequencies 0,±2π,…0,±2π,…, the estimator may exhibit significant bias.
- In practice, this bias can be mitigated by subtracting an estimate of the mean from the series before computing the finite Fourier transform.

Practical Implication:
A modified statistic can be considered to reduce mean-induced bias:

$$\widehat{d}_x^T(\lambda) = \sum_{t=0}^{T-1} h_a(t)\left[X_a(t) - \widehat{C}_a\right]e^{-i\lambda t},$$

where $\widehat{C}_a$ is a consistent estimate of C_a, such as the sample mean.

Theorem3:-
Let $X(t)$ be a strictly stationary discrete-time r-vector-valued series, and let $h_a^{(T)}(t)$ be a data window. Consider the periodogram $I_{XX}^{(T)}(\lambda)$ as defined in (2). Then, the covariance between two periodogram components is given by:

$$Cov\left[I_{a_1b_1}^{(T)}(\lambda), I_{a_2b_2}^{(T)}(\mu)\right] = \left[2\pi H_{a_1b_1}^{(T)}(0).2\pi H_{a_2b_2}^{(T)}(0)\right]^{-1} \times$$
$$\begin{Bmatrix} H_{a_1a_2}^{(T)}(\lambda-\mu)\overline{H_{b_1b_2}^{(T)}(\lambda-\mu)}f_{a_1a_2}(\lambda)f_{b_1b_2}(-\lambda) + H_{a_1b_2}^{(T)}(\lambda+\mu)\overline{H_{b_1a_2}^{(T)}(\lambda+\mu)}f_{a_1b_2}(\lambda)f_{b_1a_2}(-\lambda) \\ + T^{-1}R_T(\lambda,\mu) \end{Bmatrix},$$

where the remainder term $R_T(\lambda,\mu)$ satisfies:

$$\left|R_T(\lambda,\mu)\right| \leq K_1\left|H_a^{(T)}(\lambda)\right|\left|H_a^{(T)}(\mu)\right| + K_2\left|H_a^{(T)}(\lambda)\right|K_3\left|H_a^{(T)}(\mu)\right| + k_4,$$

for some constants K_1, K_2, K_3, K_4 and for all $a = a_1, a_2, b_1, b_2$ and $-\infty < \lambda, \mu < \infty$.

Proof:
We start from the representation:

$$I_{ab}^{(T)}(\lambda) = \frac{1}{2\pi H_{ab}^{(T)}(0)} d_a^{(T)}(\lambda)\overline{d_b^{(T)}}(\lambda)$$

The covariance between two periodogram entries can be expressed as:

$$Cov\left[I_{a_1b_1}^{(T)}(\lambda), I_{a_2b_2}^{(T)}(\mu)\right] = \frac{1}{(2\pi)^2 H_{a_1b_1}^{(T)}(0)H_{a_2b_2}^{(T)}(0)} Cov\left\{d_{a_1}^{(T)}(\lambda)\overline{d_{b_1}^{(T)}}(\lambda), d_{a_2}^{(T)}(\mu)\overline{d_{b_2}^{(T)}}(\mu)\right\}$$

Using properties of cumulants for stationary processes, this covariance decomposes into three main components:

- **Product of Covariances (Term F):**

$$Cov\left[d_{a_1}^{(T)}(\lambda), d_{a_2}^{(T)}(\mu)\right] Cov\left[\overline{d_{b_1}^{(T)}}(\lambda), \overline{d_{b_2}^{(T)}}(\mu)\right]$$

which asymptotically behaves as:

$$\left[2\pi f_{a_1a_2}^{(T)}(\lambda)H_{a_1a_2}^{(T)}(\lambda-\mu) + O(1)\right].\left[2\pi f_{b_1b_2}^{(T)}(-\lambda)H_{b_1b_2}^{(T)}(-\lambda+\mu) + O(1)\right]$$

- **Product of Cross-Covariances (Term M):**

$$Cov\left[d_{a_1}^{(T)}(\lambda), d_{a_2}^{(T)}(\mu)\right].\ Cov\left[\overline{d_{b_1}^{(T)}}(\lambda), \overline{d_{b_2}^{(T)}}(\mu)\right]$$

asymptotically:

$$\left[2\pi f_{a_1b_2}^{(T)}(\lambda)H_{a_1b_2}^{(T)}(\lambda+\mu) + O(1)\right].\left[2\pi f_{b_1a_2}^{(T)}(-\lambda)H_{b_1a_2}^{(T)}(-\lambda-\mu) + O(1)\right]$$

- **Fourth-Order Cumulant Term (Term K):**

$$Cum\left\{d_{a_1}^{(T)}(\lambda),\overline{d_{b_1}^{(T)}}(\lambda),d_{a_2}^{(T)}(\mu),\overline{d_{b_2}^{(T)}}(\mu)\right\}=(2\pi)^3 f_{a_1b_1a_2b_2}(\lambda,-\lambda,\mu)H_{a_1b_1a_2b_2}^{(T)}(0)+O(1)$$

Combining these three contributions and noting that:

$$H_{ab}^{(T)}(v)\approx T\int h_a(t)h_b(t)e^{-ivt}dt,$$

we obtain the stated expression. The remainder term $R_T(\lambda,\mu)$ collects lower-order terms and is bounded as indicated.Thus, the theorem is proved.

Conclusion:-

This paper has presented a statistical analysis of the finite Fourier transform and the periodogram in the context of discrete-time series.

Through the derivation of key asymptotic properties, we have established:

- **Distributional Results:** Under strict stationarity and appropriate windowing conditions, the finite Fourier transform converges to a multivariate normal distribution, with distinct behavior at zero and non-zero frequencies.
- **Expectation of the Periodogram:** The expected value of the periodogram was derived, revealing its relationship with the spectral density matrix and highlighting conditions under which it serves as an asymptotically unbiased estimator.
- **Covariance Structure:** The covariance between periodogram components was characterized, providing insight into the dependency structure of spectral estimates across frequencies and components.

These results contribute to the theoretical foundation for spectral estimation in multivariate time series analysis, particularly in settings involving filtered, windowed, or incompletely observed data. The findings may support further research in areas such as spectral inference, hypothesis testing, and the analysis of nonstationary or irregularly sampled time series.

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