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RESEARCH ARTICLE

GLOBAL ATTRCTIVITY AND POSITIVITY SOLUTIONS FOR NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MEASURES OF NONCOMPACTNESS

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Abstract

We prove in this paper some existence theorems concerning the attractivity and positivity of the solutions for nonlinear functional differential equations using the techniques of some measures of noncompactness. Our study is in the Banach space of real-valued functions defined, continuous and bounded on unbounded intervals together with the applications of a measure theoretic fixed point theorem of Dhage[1].Our study in this paper, it is new to the literature as regards positivity of the solutions for nonlinear functional differential equations.

Introduction:-

Nonlinear differential equations and integral equations with bounded intervals have been studied in the literature as various aspects existence, uniqueness, stability and externality of solutions. However the study of nonlinear differential and integral equations with unbounded intervals is new and exploited for the new characteristics of attractivity and asymptotic attractivity of solutions. There are two approaches for dealing with these characteristics of solutions one is classical fixed point theorems involving the hypothesis from analysis and topology, the second is the fixed point theorems involving the use of measure of noncompactness approaches has some advantages and disadvantages over the others Dhage[2,3]. In this paper, we prove some theorems on the existence and global attractivity and positivity of solutions for functional differential equations by using fixed point theorems involving the use of measures of noncompactness. Our study will be situated in the Banach space of real-valued functions which are defined, continuous and bounded on the real half axis R_+. The main tool used in our considerations is the technique of measures of noncompactness and fixed point theorem of B.C.Dhage type [1]. The assumptions imposed in our main existence theorems admit several natural realizations. These realizations are constructed with help of a certain class of sub additive functions. The results obtained in this paper generalized and extend several ones obtained earlier in a lot of papers concerning asymptotic stability of solutions for some functional integral equations [cf.1,4,5,6,7]. Our approach consists mainly in the possibility of obtaining the global attractivity, asymptotic attractivity and positivity of solutions for considered nonlinear functional Differential equations.

Auiliary Results:-

At the beginning we present some basic facts concerning the measures of noncompactness. We accept the following definitions of the concept of a measure of noncompactness given in Dhage[1].The details measures of

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noncompactness appear in Banas and Goebel[8] and the references therein. Let E be a Banach space and let $\mathcal{P}_{\mathcal{p}}(E)$be denote the class of all non-empty subsets ofE with property $\mathcal{P}$. Here $\mathcal{P}$ may be $\mathcal{P}_{cl}$ = closed, $\mathcal{P}_{bd}$ = bounded, $\mathcal{P}_{rcp}$ = relatively compact. Thus, $\mathcal{P}_{cl}(E), \mathcal{P}_{bd}(E), \mathcal{P}_{cl,bd}(E)$ and $\mathcal{P}_{rcp}(E)$denotes the classes of closed, bounded, closed and bounded and relatively compact subsets of E respectively.A function$d_H(A, B) = \max\{\sup_{a\in A} d(A, B), \sup_{b\in B} d(b, A), \}$Satisfies all the conditions of a metric on $\mathcal{P}_{bd}(E)$ is called Hausdrorff-Pompeiu metric on E, where $d(a, B) = \inf\{\|a - b\|: b \in B\}$. It is known that the hyperspace $(\mathcal{P}_{cl}(E), d_H)$ is a complete metric space. In this paper, we adopt the following axiomatic definition of the measure of noncompactness in a Banach space given by Dhage[1]. The other useful forms appear in Banas and Goebel[8] and the references therein. We need the following definitions in the sequel.

Definition:2.1. A sequence $\{A_n\}$ of non-empty sets in $\mathcal{P}_{bd}(E)$ is said to converges to a set A, called the limiting set if $d_H(A_n, A) \to 0$ as $n \to \infty$. A mapping $\mu: \mathcal{P}_{bd}(E) \to \mathbb{R}^+$ is called continuous if for any sequence $\{A_n\}$ in $\mathcal{P}_{bd}(E)$ we have

d_H (A_n, A) → 0 ⇒ |μ(A_n) − μ(A) | → 0asn → ∞.

Definition:2.2. A mapping $\mu: \mathcal{P}_{bd}(E) \to \mathbb{R}^+$ is said to be nondecreasing if $A, B \in \mathcal{P}_{bd}(E)$ are any two sets with $A \subseteq B$, then $\mu(A) \le \mu(B)$, where$\le$ is a order relation by inclusion in $\mathcal{P}_{bd}(E)$.

Definition:2.3. A function $\mu: \mathcal{P}_{bd}(E) \to \mathbb{R}^+$ is called a measure of noncompactness if it satisfies

i. $\phi \neq \mu^{-1}(0) \subset \mathcal{P}_{rcp}(E)$,
ii. $\mu(\overline{A}) = \mu(A)$, where $\overline{A}$ denotes the closure of A,
iii. $\mu(\text{conv}A) = \mu(A)$, where convA denotes the convex hull of A,
iv. μ is nondecreasing, and
v. If $\{A_n\}$ is a decreasing sequence of sets in $\mathcal{P}_{bd}(E)$ such tha $\lim_{n\to\infty}\mu(A_n) = 0$, then the limiting set $A_\infty = \lim_{n\to\infty} A_n$ is non-empty

The family kerμ described in (i) is said to be the kernel of the measure of noncompactness μ and $\ker\mu = \{A \in \mathcal{P}_{bd}(E): \mu(A) = 0\} \subseteq \mathcal{P}_{rcp}(E)$. The measureμ is called complete if the kerμ of μ consists of all possible relatively compact subsets of E.

The measureμ is called sublinear if it satisfies

vi. $\mu(\lambda A) = |\lambda|\mu(A)$ for $\lambda \in \mathbb{R}$, and
vii. $\mu(A + B) \le \mu(A) + \mu(B)$.

There do exist the sublinear measures of noncompactness on Banach space E. Observe that the limiting set A_∞ from (v) is a member of family kerμ. In facts, science $\mu(A_\infty) \le \mu(A_n)$ for any n, we infer that $\mu(A_\infty) = 0$. There fore $A_\infty \in \ker\mu$.

Definition:2.4. A mapping $Q: E \to E$ is called $D-\text{set}-\text{Lipschitz}$ if there exists a continuous nondecreasing function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(Q(A)) \le \phi(\mu(A))$ for all $A \in \mathcal{P}_{bd}(E)$ with $Q(A) \in \mathcal{P}_{bd}(E)$, where $\phi(0) = 0$. Sometimes we call the function ϕ to be D-function of Q on E. When $\phi(r) = kr, k > 0$ then Q is called a K-set contraction on E. Further if $\phi(r) < r$ for $r > 0$, then Q is called a nonlinear D-set contraction on E.

Theorem:2.1(Dhage[1]): Let C be a non-empty, closed, convex and bounded subset of a Banach space E, and let $Q: C \to C$ be a continuous and nonlinear D-set contraction. Then Q has a fixed point.

Remark.2.1: Let Fix(Q) denote the set of all fixed points of the operator Q which belong to c. It can be shown in theorem.2.1 $\text{Fix}(Q) \in \ker\mu$. In fact if $\text{Fix}(Q) \notin \ker\mu$, then $\mu(\text{Fix}(Q)) > 0$ and $Q(\text{Fix}(Q)) = \text{Fix}(Q)$. Now from nonlinear D-set contraction, $\mu\left(Q(\text{Fix}(Q))\right) \le Q\left(\mu(\text{Fix}(Q))\right)$ This is a contradiction.

Since $\phi(r) < r$ for $r > 0$. Hence $\text{Fix}(Q) \in \ker\mu$. Our further considerations will be placed in Banach space $BC(\mathbb{R}_+, \mathbb{R})$ with standard supremum norm

$$\|x\| = \sup\{|x(t)|: t \in \mathbb{R}_+\}$$

for our purpose we will use the Hausdorff measure of noncompactness in $BC(\mathbb{R}_+, \mathbb{R})$ and is defined as follows. Let us fix a nonempty and bounded subset X of the space $BC(\mathbb{R}_+, \mathbb{R})$ and positive number T. For $x \in X, \epsilon \ge 0$ denote by

$$\omega^T(x, \epsilon) = \sup\{|x(t) - x(s)|: t, s \in [0, T], |t - s| \le \epsilon\}$$

Next, let us put

$$\omega^T(X, \epsilon) = \sup\{\omega^T(x, \epsilon): x \in X\}$$
$$\omega_0^T(X) = \lim_{\epsilon\to 0} \omega^T(X, \epsilon).$$

It is known that ω_0^T is a measure of noncompactness in the Banach space $C([0,T],\mathbb{R})$ of continuous and real-valued functions defined on a closed and bounded interval $[0,T]$ in $\mathbb{R}$ which is equivalent to Hausdroff or ball measure on noncompactness in it. Now one has

$$\chi(X) = \frac{1}{2}\omega_0^T(X)$$

For any bounded subset χ of $C([0,T],\mathbb{R})$ see Banas and Goebel [3] and the reference therein. We define

$$\omega_o(X) = \lim_{T\to\infty} \omega_0^T(X)$$

Now, for a fixed number $t \in \mathbb{R}_+$, let us denote $X(t) = \{x(t): x \in X\}$,
$\|X(t)\| = sup\{|x(t)|: x \in X\}$.
and
$\|X(t) - c\| = sup\{|x(t) - c|: x \in X\}$.
Let us consider the function μ defined on the family $\mathcal{P}_{bd}(X)$ by
s$\mu_a(X) = max\{\omega_0(X), lim_{t\to\infty} sup\, diam X(t)\}$,
$\mu_b(X) = max\{\omega_0(X), lim_{t\to\infty} sup\|X(t)\|\}$,
and $\mu_c(X) = max\{\omega_0(X), lim_{t\to\infty} sup\|X(t) - c\|\}$.
For any bounded subset X of $BC(\mathbb{R}_+,\mathbb{R})$ define
$\delta(X) = sup\{lim_{t\to\infty} sup(|x(t)| - x(t)) : x \in X\}$.
Define the functions $\mu_{ad},\ \mu_{bd},\ \mu_{cd}: \mathcal{P}_{bd}(E) \to \mathbb{R}_+$ by
$\mu_{ad}(X) = max\{\mu_a(X), \delta(X)\}$ 2.2
$\mu_{bd}(X) = max\{\mu_b(X), \delta(X)\}$ 2.3
$\mu_{cd}(X) = max\{\mu_c(X), \delta(X)\}$ 2.4
for all $X \in \mathcal{P}_{bd}(E)$
It can be shown as in Banas[4] that the functions $\mu_a,\ \mu_b,\ \mu_c, \mu_{ad},\ \mu_{bd}\ and \mu_{cd}$ are measures of noncompactness in the space$BC(\mathbb{R}_+,\mathbb{R})$. The $ker\mu_a, ker\mu_b and ker\mu_c$ of the measures $\mu_a, \mu_b and \mu_c$ consist of non empty and bounded subsets X are locally equicontinuous on $\mathbb{R}_+$.

In order to introduce further concepts used in this article, let us assume that $E = BC(\mathbb{R}_+,\mathbb{R})$ and let Ω be a subset of X. Let $Q: E \to E$ be a operator and consider the following operator equation in E,
$Qx(t) = x(t)$ 2.5
For all $t \in \mathbb{R}_+$. Below we give different characterizations of the solutions for the operator (2.5) on $\mathbb{R}_+$.

Definition:2.5. We say that solutions of equation (2.5) are locally attractive if there exists a closed ball $\overline{\mathscr{B}}_r(x_o)$ in space $BC(\mathbb{R}_+,\mathbb{R})$ for some $x_o \in BC(\mathbb{R}_+,\mathbb{R})$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (2.5) belonging to $\overline{\mathscr{B}}_r(x_o) \cap \Omega$. we have
$lim_{t\to\infty}(x(t) - y(t)) = 0$ 2.6
In the case when the limit (2.3) is uniform with respect to the set $\overline{\mathscr{B}}_r(x_o) \cap \Omega$ i.e. when for each $\epsilon > 0, \exists T > 0$ such that
$|x(t) - y(t)| \leq \epsilon$ 2.7
for all $x, y \in \overline{\mathscr{B}}_r(x_o) \cap \Omega$ being solutions of (2.1) and for$t \geq T$, we will say that solutions of (2.5) are uniformly locally attractive on$\mathbb{R}_+$.
Definition:2.6. The solution $x = x(t)$ of equation (2.5) is said to be globally attractive if (2.7) holds for each solution$y = y(t)$ of (2.5) on Ω. In the case when the condition (2.9) is satisfied uniformly with respect to the setΩ i.e. if for every $\epsilon > 0, \exists T > 0$ such that the inequality (2.7) is satisfied for all $x, y \in \Omega$ being the solution of (2.5) and
$t \geq T$, we will say that solutions of the equation (2.5) are uniformly globally attractive on$\mathbb{R}_+$.

The following definitions appear in Dhage[2]
Definition:2.7. A line $y(t) = c$ where c a real number is called a attractor for a solution$x \in BC(\mathbb{R}_+,\mathbb{R})$ to the equation (2.5) if $lim_{t\to\infty}(x(t) - c) = 0$. In such case the solution x to the equation (2.6) is called to be asymptotic to the line $y(t) = c$ and the line is asymptote for the solution$x$ on $\mathbb{R}_+$.
Let us mention that the concepts of global attractivity of solutions are recently introduced in Hu and Yan[7] while the concepts of local and global asymptotic attractivity have been presented in Dhage[2]. Similarly, the concepts of uniform local and global attractivity were introduced in Banas and Rzepka[5].

Next we introduce the new concept of local and global asymptotic positivity of solution for equation2.5) in $\mathcal{BC}(\mathbb{R}_+,\mathbb{R})$.

Definition:2.8. A solution$x$ of equation (2.5) is called locally ultimately positive if there exist a closed ball $\overline{\mathcal{B}}_r(x_o)$ in $BC(\mathbb{R}_+,\mathbb{R})$ for some $x \in BC(\mathbb{R}_+,\mathbb{R})$ such that $x \in \overline{\mathcal{B}}_r(x_o)$ and

$$\lim_{t\to\infty}[|x(t)| - x(t)] = 0 \qquad 2.8$$

When for each $\epsilon > 0, \exists T > 0$ such that

$$\big||x(t)| - x(t)\big| \le \epsilon \qquad 2.9$$

For all x being solutions of (2.5) and for $t \ge T$, we will say that solutions of equations (2.5) are uniformly locally ultimately positive on $\mathbb{R}_+$.

Definition:2.9: A solution $x \in C(\mathbb{R}_+,\mathbb{R})$ of equation (2.5) is called globally ultimately positive if equation (2.9) is satisfied. In this case when the limit (2.8) is uniform with respective to the solution set of the operator equation (2.5) in $C(\mathbb{R}_+,\mathbb{R})$. i.e. when for each $\epsilon > 0, \exists T > 0$ such that (2.9) is satisfied for all x being solutions of equations of (2.5) and for $t \ge T$, we will say that solutions of equations (2.5) are uniformly globally ultimately positive on $\mathbb{R}_+$. In the following section we prove the main results of this article.

Attrctivity And Positivity Solution:-

Let $\mathbb{R}$ be the real line and let $\mathbb{R}_+$ be the set of non negative real numbers. Consider the functional differential equation (in short FDE)

$$\frac{d}{dt}\left[\frac{x(t)}{f\big(t,x(\alpha(t))\big)}\right] = g\Big(t, x\big(\gamma(t)\big)\Big) \qquad 3.1$$

for $t \in \mathbb{R}_+$, where $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$, and $\alpha,\gamma:\mathbb{R}_+\to\mathbb{R}_+$.

By a solution of the FDE (3.1) we mean a function in $C(\mathbb{R}_+,\mathbb{R})$ that satisfies the equation (3.1), where $C(\mathbb{R}_+,\mathbb{R})$ is the space of continuous real-valued functions defined on $\mathbb{R}_+$. For $t \in \mathbb{R}_+$, the FDE (3.1) reduces to the functional integral equation (in short FIE)

$$x(t) = q(t) + f\Big(t, x\big(\alpha(t)\big)\Big) + \int_0^{\beta(t)} g\Big(t, x\big(\omega(s)\big)\Big)\, ds \qquad 3.2$$

where : $\mathbb{R}_+\to\mathbb{R}$, $\beta:\mathbb{R}_+\to\mathbb{R}_+$.

The type of integral equation (3.2) has been studied in Dhage[3] and references given therein. For global attractivity of solutions via classical hybrid fixed point theory observe that the type of above integral equation (3.2) includes several classes of functional, integral and functional integral equations considered in the literature (cf[1,4,5,6,7] and references therein). Let us also mention that the following type of functional integral equation considerd in Banas and Dhage[6],

$$x(t) = f\Big(t, x\big(\alpha(t)\big)\Big) + \int_0^{\beta(t)} g\Big(t, s, x\big(\omega(s)\big)\Big)\, ds \qquad 3.3$$

is also special case of the equation (3.2) which further includes the functional integral equation considered in Banas and Rzepk[5] where $\alpha(t) = \beta(t) = \gamma(t), t \in \mathbb{R}_+$. Therefore FIE(3.2) means FDE(3.1) is more general and so the attractivity and positivity of this paper include the attractivity and positivity results for all the above mentioned integral equations which are also new to the literature.

The equation (3.2) will be considered under the following assumptions.

(A_0) The functions $\alpha,\beta,\gamma:\mathbb{R}_+\to\mathbb{R}_+$ are continuous and satisfy $t \le \alpha(t)$ for $t \in \mathbb{R}_+$.

(A_1) The function $q:\mathbb{R}_+\to\mathbb{R}$ is continuous and bounded.

(A_2) The function $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a bounded function $\ell:\mathbb{R}_+\to\mathbb{R}$ with bound L and a positive constant M such that

$$|f(t,x) - f(t,y)| \le \frac{\ell(t)\max\{|x-y|\}}{M + \max\{|x-y|\}}$$

for $t \in \mathbb{R}_+$ and for $x, y \in \mathbb{R}$. Moreover, we assume that $L \le M$.

(A_3) The function $t \to f(t,o)$ is bounded on $\mathbb{R}_+$ with $F_0 = \sup\{|f(t,o)| : t \in \mathbb{R}_+\}$.

(A_4) The function $g:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a continuous function $b:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ such that $|g(t,s,x)| \le b(t,s)$ for $t,s \in \mathbb{R}_+$.

Moreover, we assume that $\lim_{t\to\infty}\int_0^{\beta(t)} b(t,s)\,ds = 0$.

Remark.3.1: Hypothesis (A_2) is satisfied if the function and satisfied the condition,

$$|f(t,x) - f(t,y)| \le \frac{\ell(t)|x-y|}{2M + |x-y|} \qquad 3.4$$

for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$, where $L \le M$, and the function ℓ is defined as in hypothesis (A_2) which further yields the usual Lipschitz condition on the function f,

$$|f(t,x) - f(t,y)| \le \frac{\ell(t)}{2M}|x-y| \qquad 3.5$$

for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$ provided $L < M$. Our hypothesis (A_2) is more general that existing in the literature.

We will proceed for our main results.

Theorem:3.1: Under the above assumptions (A_0)- (A_4), FDE (3.1) has at least one solution in the space $BC(\mathbb{R}_+, \mathbb{R})$. Moreover, solutions of the equation FDE (3.1) are globally uniformly attractive on $\mathbb{R}_+$.

Proof: Consider the operator Q defined on the space $BC(\mathbb{R}_+, \mathbb{R})$ be the formula

$$Qx(t) = q(x) + f\left(t, x\left(\alpha(t)\right)\right) + \int_0^{\beta(t)} g\left(t, s, x\left(\alpha(s)\right)\right) ds \qquad 3.6$$

Observe that for any $x \in BC(\mathbb{R}_+, \mathbb{R})$ the function Qx is continuous on $\mathbb{R}_+$. Moreover for any fixed $t \in \mathbb{R}_+$ we obtain

$$|Qx(t)| \leq |q(x)| + \left|f\left(t, x\left(\alpha(t)\right)\right)\right| + \int_0^{\beta(t)} \left|g\left(t, s, x\left(\alpha(s)\right)\right)\right| ds$$

$$\leq |q(x)| + \left|f\left(t, x\left(\alpha(t)\right)\right) - f(t, 0)\right| + |f(t, 0)| + \int_0^{\beta(t)} b(t, s) ds$$

$$\leq \|q\| + \frac{L\max\left\{\left|x\left(\alpha(t)\right)\right|\right\}}{M + \max\left\{\left|x\left(\alpha(t)\right)\right|\right\}} + |f(t, 0)| + \int_0^{\beta(t)} b(t, s) ds$$

$$\leq \|q\| + \frac{L\|x\|}{M + \|x\|} + F_0 + v(t)$$

$$\leq \|q\| + \frac{L\|x\|}{M + \|x\|} + F_0 + V$$

where $v(t) = \int_0^{\beta(t)} b(t, s) ds$, $V = \sup\ \{v(t): t \in \mathbb{R}_+\}$ is finite by (A_4).

From the above estimate we deduce that

$$\|Q\| \leq \|q\| + L + F_0 + V \qquad 3.7$$

for all $x \in BC(\mathbb{R}_+, \mathbb{R})$. This means that the operator Q transforms the space $BC(\mathbb{R}_+, \mathbb{R})$ into itself from (3.7) the operator Q transforms continuously the space $BC(\mathbb{R}_+, \mathbb{R})$ into the closed ball $\overline{\mathcal{B}}_r(0)$, where $r = \|q\| + L + F_0 + V$. Because of this fact, the existence of solutions for the FDE (3.1) is global in nature.

We will consider the operator Q as a mapping from $\overline{\mathcal{B}}_r(0)$ into itself. New we show that the operator Q is continuous on the ball $\overline{\mathcal{B}}_r(0)$. Let $\epsilon > 0$ and take $x, y \in \overline{\mathcal{B}}_r(0)$ such that $\|x - y\| < \epsilon$. Then we get

$$|Qx(t) - Qy(t)| \leq \left|f\left(t, x\left(\alpha(t)\right)\right) - f\left(t, y\left(\alpha(t)\right)\right)\right|$$

$$+ \int_0^{\beta(t)} \left|g\left(t, s, x\left(\alpha(s)\right)\right) - g\left(t, s, y\left(\alpha(s)\right)\right)\right| ds$$

$$\leq \frac{L\max\left\{\left|x\left(\alpha(t)\right) - y\left(\alpha(t)\right)\right|\right\}}{M + \max\left\{\left|x\left(\alpha(t)\right) - y\left(\alpha(t)\right)\right|\right\}} + \int_0^{\beta(t)} \left[\left|g\left(t, s, x\left(\alpha(s)\right)\right)\right| + \left|g\left(t, s, y\left(\alpha(s)\right)\right)\right|\right] ds$$

$$\leq \frac{L\|x - y\|}{M + \|x - y\|} + 2\int_0^{\beta(t)} b(t, s) ds$$

$\leq \epsilon + 2v(t)$.

Hence, in virtue of assumption (A_4) we infer that there exists $T > 0$ such that $v(t) \leq \epsilon$ for $t \geq T$. Thus for $t \geq T$ from (3.3) we derive that

$$|Qx(t) - Qy(t)| \leq 3\epsilon \qquad 3.8$$

Further let us assume that $t \in [0, T]$ then evaluating similarity s above we get

$$|Qx(t) - Qy(t)| \leq \epsilon + \int_0^{\beta(t)} \left|g\left(t, s, x\left(\alpha(s)\right)\right) - g\left(t, s, y\left(\alpha(s)\right)\right)\right| ds$$

$$\leq \epsilon + \int_0^{\beta(t)} \omega_r^T(g, \epsilon) ds$$

$$\leq \epsilon \beta_T \omega_r^T(g,\epsilon)$$

Where $\beta_T = sup\ \{\beta(t): t \in [0,T]\}$ and

$$\omega_r^T(g,\epsilon) = sup\ \{|g(t,s,x) - g(t,s,y)|: t \in [0,T], s \in [0,\beta_T], x,y \in [-r,r], |x-y| \leq \epsilon\}$$

3.10

Obviously we have that $\beta_T < \infty$. Moreover from the uniform continuity of the function $g(t,s,x)$ on the set $[0,T] \times [0,\beta_T] \times [-r,r]$. we derive that $\omega_r^T(g,\epsilon) \to 0$ as$\epsilon \to 0$. Now from (3.9),(3,10) and above established facts we conclude that the operator Q maps continuously the closed ball $\overline{\mathcal{B}}_r(0)$ into itself.

Further on let us take nonempty subset X of the ball $\overline{\mathcal{B}}_r(0)$. Next $T > 0$ and $\epsilon > 0$, let us choose $x \in X$ and $t_1, t_2 \in [0,T]$ with $|t_1 - t_2| \leq \epsilon$. Without loss of generality we may assume that $t_1 < t_2$. Then taking into account our assumptions, we get

$$|(Qx)(t_2) - (Qx)(t_1)| \leq |q(t_2) - q(t_1)| + \left|f\left(t_2, x\left(\alpha(t_2)\right)\right) - f\left(t_1, x\left(\alpha(t_1)\right)\right)\right|$$

$$+ \left|\int_0^{\beta(t_2)} g\left(t_2, s, x\left(\alpha(s)\right)\right) ds - \int_0^{\beta(t_2)} g\left(t_1, s, x\left(\alpha(s)\right)\right) ds\right|$$

$$+ \left|\int_0^{\beta(t_2)} g\left(t_1, s, x\left(\alpha(s)\right)\right) ds - \int_0^{\beta(t_1)} g\left(t_1, s, x\left(\alpha(s)\right)\right) ds\right|$$

$$\leq \omega^T(q,\epsilon) + \frac{Lmax\ \{|x(\alpha(t_2)) - x(\alpha(t_1))|\}}{M + max\ \{|x(\alpha(t_2)) - x(\alpha(t_1))|\}} + \omega_r^T(f,\epsilon)$$

$$+ \int_0^{\beta(t_2)} \left|g\left(t_1, s, x\left(\alpha(s)\right)\right) - g\left(t_2, s, x\left(\alpha(s)\right)\right)\right| ds$$

$$+ \left|\int_{\beta(t_1)}^{\beta(t_2)} \left|g\left(t, s, x\left(\alpha(s)\right)\right)\right| ds\right|$$

$$\leq \omega^T(q,\epsilon) + \frac{Lmax\ \{|\omega^T(x,\omega^T(\alpha,\epsilon))|\}}{M + max\ \{|\omega^T(x,\omega^T(\alpha,\epsilon))|\}} + \omega_r^T(f,\epsilon) + \int_0^{\beta_T} \omega_r^T(g,\epsilon) ds + \omega^T(\beta,\epsilon) G_T^r \quad 3.11$$

Where $\omega_r^T(q,\epsilon) = sup\ \{|q(t_2) - q(t_1)|: t_1, t_2 \in [0,T], |t_1 - t_2| \leq \epsilon\}$

$$\omega_r^T(f,\epsilon) = sup\ \{|f(t_2,x) - f(t_1,x)|: t_1, t_2 \in [0,T], |t_1 - t_2| \leq \epsilon, x, y \in [-r,r]\}$$

$$\omega_r^T(g,\epsilon) = sup\ \left\{\begin{matrix} |g(t_2,s,x) - g(t_1,s,x)|: t_1, t_2 \in [0,T], |t_1 - t_2| \leq \epsilon, \\ s \in [0,\beta_T], x, y \in [-r,r] \end{matrix}\right\}$$

$$G_T^r = sup\ \{|g(t,s,x)|: t \in [0,T], s \in [0,\beta_T], x \in [-r,r]\}.$$

from the above estimate we derive the following

$$\omega^T(\mathrm{Q}x,\epsilon) \leq \omega^T(q,\epsilon) + \frac{Lmax\ \{|\omega^T(x,\omega^T(\alpha,\epsilon))|\}}{M + max\ \{|\omega^T(x,\omega^T(\alpha,\epsilon))|\}} + \omega_r^T(f,\epsilon) + \int_0^{\beta_T} \omega_r^T(g,\epsilon) ds + \omega^T(\beta,\epsilon) G_T^r \quad 3.12$$

Observe that $\omega^T(q,\epsilon) \to 0, \omega_r^T(f,\epsilon) \to 0$ and $\omega_r^T(g,\epsilon) \to 0$ as $\epsilon \to 0$, which is a simple consequence of the uniform continuity of the functions q, f, g on the set $[0,T]$,
$[0,T] \times [-r,r]$ and $[0,T] \times [0,\beta_T] \times [-r,r]$ respectively. Moreover it is obvious that the constant G_T^r is finite and $\omega^T(\alpha,\epsilon) \to 0, \omega^T(\beta,\epsilon) \to 0$ as $\epsilon \to 0$. Thus linking the established facts with the estimate (3.12) we get,

$$\omega_0^T(Qx) \leq \frac{L\,\omega_0^T(X)}{M + \omega_0^T(X)} \quad 3.13$$

Now, taking into account our assumptions, for fixed $t \in \mathbb{R}_+$ and for $x, y \in X$ we deduce the following

$$|Qx(t) - Qy(t)| \leq \frac{Lmax\ \{|x(\alpha(t)) - y(\alpha(t))|\}}{M + max\ \{|x(\alpha(t)) - y(\alpha(t))|\}} +$$

$$\int_0^{\beta(t)} \left[\left|g\left(t, s, x\left(\gamma(s)\right)\right)\right| + \left|g\left(t, s, y\left(\gamma(s)\right)\right)\right|\right] ds$$

$$\leq \frac{L\max\{|x(\alpha(t)) - y(\alpha(t))|\}}{M + \max\{|x(\alpha(t)) - y(\alpha(t))|\}} + 2v(t)$$

$$\leq \frac{L\max\{diam X(\alpha(t))\}}{M + \max\{diam X(\alpha(t))\}} + 2v(t)$$

Hence we obtain

$$diam(Qx)(t) \leq \frac{L\max\{diam X(\alpha(t))\}}{M + \max\{diam X(\alpha(t))\}} + 2v(t)$$

In view of assumptions (A_0) and (A_4) yields

$$\lim_{t\to\infty} \sup daim (Qx)(t) \leq \frac{L \lim_{t\to\infty} \sup \max\{diam X(\alpha(t))\}}{M + \lim_{t\to\infty} \sup \max\{diam X(\alpha(t))\}}$$

$$\leq \frac{L \lim_{t\to\infty} \sup diam X(t)}{M + \lim_{t\to\infty} \sup diam X(t)} \quad 3.14$$

Further using the measure of noncompactness μ_a defined by the (2.2) and keeping in mind the estimate (3.13) and (3.14), we get

$$\mu_a(QX) = \max\left\{\omega_0(Qx), \lim_{t\to\infty} \sup diam QX(t)\right\}$$

$$\leq \max\left\{\frac{L\,\omega_0(X)}{M + \omega_0(X)}, \frac{L \lim_{t\to\infty} \sup diam X(t)}{M + \lim_{t\to\infty} \sup diam X(t)}\right\}$$

$$\leq \frac{L\max\left\{\omega_0(X), \lim_{t\to\infty} \sup diam QX(t)\right\}}{M + \max\left\{\omega_0(X), \lim_{t\to\infty} \sup diam QX(t)\right\}}$$

$$= \frac{L\mu_a(X)}{M+\mu_a(X)} \quad 3.15$$

Since $L \leq M$ by of assumption (A_2) from the above estimate, $\mu_a(QX) \leq \phi(\mu_a(X))$ where $\phi(r) = \frac{Lr}{M+r} < r$ for $r > 0$. Hence we yield theorem (2.1) to deduce that the operator Q has a fixed point x in the ball $\overline{\mathscr{B}}_r(0)$. Obviously x is solution of the FIE (3.2) means solution of FDE (3.1). Moreover taking into account that the image of the space $BC(\mathbb{R}_+, \mathbb{R})$ under the operator $Q$is contained in the ball $\overline{\mathscr{B}}_r(0)$ we infer that the set Fix(Q) of all fixed points of $Q$is contained in $\overline{\mathscr{B}}_r(0)$. Obviously, the set Fix$(Q)$ of all contains all solutions of the FIE (3.2) means FDE (3.1). From remark (2.1) the set Fix(Q) belongs to the family kerμ_a. Now, taking into account the description of sets belonging to kerμ_a we deduce that all solutions for the FIE(3.2) are globally uniformly attractive on $\mathbb{R}_+$. This completes the proof.

Remark:3.2: When $q = 0, f(t,x)$ and $g(t,s,x)$ in our theorem 3.1 we obtain the global attractivity result for the FDE(3.1). Note that the global attractivity result for (3.3) is also proved in Banas and Dhage[6] under the same hypothesis, but under the stronger hypothesis of (A_2)that$L < M$. Therefore, our theorem 3.1 generalize and improve the existence results of Dhage[3] and Banas and Dhage[6] and thereby the results of Banas and Rezpka[5] under weaker conditions with a new measure of noncompactness in the Banach space $BC(\mathbb{R}_+, \mathbb{R})$.

To prove next result concerning the asymptotic positivity of the attractive solution we need the following hypothesis in the sequel.

(A_5) The functions q and f satisfy

$\lim_{t\to\infty}[|q(t)| - q(t)] = 0$ and $\lim_{t\to\infty}[|f(t,x)| - f(t,x)] = 0$ for all $x \in \mathbb{R}_+$.

Theorem:3.2: Under the hypotheses of theorem 3.1 and (A_5), the FDE (3.1) has at least one solution on $\mathbb{R}_+$. Moreover, solutions of the FDE(3.1) are uniformly globally attractive and ultimately positive on $\mathbb{R}_+$.

Proof: Consider the closed ball$\overline{\mathscr{B}}_r(0)$ in the Banach space $BC(\mathbb{R}_+, \mathbb{R})$, where the real number r is given as in the proof of theorem 3.1 and define a mapping $Q: BC(\mathbb{R}_+, \mathbb{R}) \to BC(\mathbb{R}_+, \mathbb{R})$ by (3.7). Then it is shown as in the proof of theorem 3.1that Q defines a continuous mapping from the space $BC(\mathbb{R}_+, \mathbb{R})$ into ball$\overline{\mathscr{B}}_r(0)$. In particular, Qmaps $\overline{\mathscr{B}}_r(0)$ into itself. Next we show that Q is a nonlinear-set-contraction with respective to the measure μ_{ad} of noncompactness in Banach space$BC(\mathbb{R}_+, \mathbb{R})$. We know that for any $x \in \mathbb{R}$.

Now for any $x \in \overline{\mathscr{B}}_r(0)$, one has

$$||Qx(t)| - Qx(t)| \leq ||q(t)| - q(t)| + \left||f(t, x(\alpha(t)))| - f(t, x(\alpha(t)))\right|$$

$$+\int_0^{\beta(t)} \left[\left|g\left(t,s,x(\gamma(s))\right)\right| - g\left(t,s,x(\gamma(s))\right)\right] ds$$

$$\leq \big||q(t)| - q(t)\big| + \left|\left|f\left(t,x(\alpha(t))\right)\right| - f\left(t,x(\alpha(t))\right)\right| + 2v(t).$$

Taking the limit supremum over t, we have

$$\lim_{t\to\infty}\sup \big||Qx(t)| - Qx(t)\big|$$

$$\leq \lim_{t\to\infty}\sup \big||q(t)| - q(t)\big| + \lim_{t\to\infty}\sup \left|\left|f\left(t,x(\alpha(t))\right)\right| - f\left(t,x(\alpha(t))\right)\right| + 2\lim_{t\to\infty}\sup v(t)$$

$$= 0$$

for all $x \in \overline{\mathscr{B}}_r(0)$. This implies that $\delta(Qx) = 0$ for all subsets X of $\overline{\mathscr{B}}_r(0)$. Further, using the measure of noncompactness μ_a defined by the formula (2.2) and keeping in mind the estimates (3.13) and (3.14), we obtain

$$\mu_{ad}(QX) = \max\{\mu_{ad}(QX), \delta(Qx)\}$$

$$\leq \max\left\{\frac{L\mu_a(X)}{M+\mu_a(X)}, 0\right\}$$

$$= \frac{L\mu_a(X)}{M+\mu_a(X)}$$

$$\leq \frac{L\mu_{ad}(X)}{M+\mu_{ad}(X)}$$

Since $L \leq M$ in view of assumption (A_2), from the above estimate we infer that $\mu_{ad}(QX) \leq \phi(\mu_{ad}(X))$, where $\phi(r) = \frac{Lr}{M+r} < r$ for $r > 0$. Hence we apply theorem 2.2 to deduce that the operator Q has a fixed point x in the ball $\overline{\mathscr{B}}_r(0)$. Obviously x is a solution of the FDE (3.1). Moreover, taking into account that the image of the space $BC(\mathbb{R}_+, \mathbb{R})$ under the operator Q is contained in the ball $\overline{\mathscr{B}}_r(0)$ we infer that the set Fix (Q) of all fixed points of Q is contained in $\overline{\mathscr{B}}_r(0)$. Obviously, the set $Fix(Q)$ contains all solutions of all the equation (3.1). On the other hand, from remark 2.1 we conclude that the set $Fix(Q)$ belongs to the family $\ker \mu_{ad}$ we deduce that all solutions of the equation (3.1) are uniformly globally attractive and positive on $\mathbb{R}_+$. This completes the Proof.

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