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RESEARCH ARTICLE

COMPREHENSIVE REVIEW OF DOUBLE INTEGRAL TRANSFORMS

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Abstract

Integral transform techniques play a fundamental role in obtaining analytical and semi-analytical solutions of differential, integral, and integro-differential equations arising in science and engineering. In recent years, several hybrid (double) integral transforms have been introduced to enhance the efficiency and applicability of classical transform methods. This paper presents a comprehensive comparative study of selected double integral transforms, the study systematically compares the transforms of standard elementary functions and examines their operational properties, including linearity, differentiation, initial and boundary value properties.The analysis highlights the mathematical similarities and distinctive characteristics of each transform, providing a unified framework for evaluating their computational effectiveness. Furthermore, the paper discusses the relative advantages of these transforms with respect to simplifying complex mathematical models, reducing computational effort, and improving the solution process for ordinary and partial differential equations. Their applicability is explored across diverse domains, including mathematical physics, heat and mass transfer, fluid dynamics, control engineering, signal processing, electromagnetics, quantum mechanics, population dynamics, and engineering sciences. The comparative investigation demonstrates that although these hybrid transforms possess analogous operational structures, each offers specific advantages depending on the nature of the governing equations and initial or boundary conditions. The results presented in this study provide valuable guidance for researchers in selecting the most appropriate double integral transform for analytical modelling and contribute to the ongoing development of transform-based solution methodologies in applied mathematics and interdisciplinary research.

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Introduction:-

Integral transforms provide a unique mathematical technique which helps us to convert analytical complex problems into simple algebraic ones. Based on unique features of each integral transform, their double integral transformed proposed with their wide applications in different areas of the sciences and technology. One-dimensional wave equation, one-dimensional heat flow equation is solved by using these double Fourier-Laplace Transform by V. D. Sharma and A.N. Rangari [6]. Sedeeg, A.K.; Mahamoud, Z.I.; Saadeh, R. presented solution of homogeneous Laplace equation, homogeneous Telegraph equation, homogeneous heat equation, Klein-Gordon equation, advection-diffusion equation. The Goursat equation with the help of double Laplace -ARA transform [7]. Raed R. Abu Awwad, Monther AI-Momani, Baha Abughaleh, Ali Jaradat, Abdul Karim Farah presented application of Sumudu-Sawi transform for finding solution of Klein -Gordon equation, telegraph equation, Voltera-integroparial differential equation [8]. Double ARA-Sumudu transform used to solve wave equation, telegraph equation, Klein-Gordon equation by Rania Saadeh, Ahmad Qaa and Aliaa Burqan [9]. ARA-Sawi Transform its application to partial differential equation demonstrated by Raed R. Abu Awwad, Monther Al-Momani, Ali -Jaradat, Baha Abughaaleh, Ahmad AI-Natoor [10].

Application of Combined Sumudu-Shehu transform discussed by Monther AI-Momani, Ali Jaradat, Baha Abughazaleh, Abdul Karim Farah [11]. Monther AI-Momani, Ali Jaradat, Baha Abughazaleh also presented application of double Laplace-Sawi transform to solve partial differential equations. The remainder of this paper is structured as follows: section 2 we have given definitions of integral transforms which show analysis of kernel. Section 3 presents definition and properties of double Integral transforms and also their derivative formulas, Section 4 contains key theoretical and behavioral characteristics of integral transform. Section 5 highlights Special Advantages and Section 6 reviews the applications of these transforms in diverse areas of applied mathematics, engineering, and the physical sciences, illustrating their significance in the analysis of ordinary and partial differential equations, mathematical physics, heat transfer, fluid mechanics, signal processing, control systems, and related disciplines. Finally, the paper concludes with a comprehensive comparative assessment of the selected double integral transforms, providing useful insights into their theoretical foundations, practical relevance, and potential directions for future research.

Preliminaries and Kernel Analysis:

Let A be the set of functions $f(t)$ defined for $t \geq 0$ which are of exponential order, meaning $|f(t)| \leq M\, e^{kt}$ for some constant M, k.

Now here we define some double integral transforms under review.

Laplace Transform:

Laplace transform of function $f(t)$ is defined as

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t \,, \quad Re\ s > 0$$

Fourier Transform:

The Fourier transform was invented by French mathematician and Physicist Jean-Baptiste Joseph Fourier in the early 19th century. He first presented his groundbreaking ideas in 1807 and later published them in his 1822 workbook.

Fourier transform of function $f(t)$ is defined as

$$F\{f(t)\} = \int_{-\infty}^{\infty} e^{-iwt} f(t) \ \mathrm{d}t$$

The Sumudu Transform :-

In the early 1990s, new integral transform is introduced by Sri Lankan Engineer and mathematician G.K. Watugala while working at the department of Electrical and Electronics Engineering at the University of Peradeniya, Sri Lanka , which preserve Unit property.

Sumudu transform of function $f(t)$ is defined as,

$$S\{f(t)\} = \frac{1}{k}\int_0^{\infty} e^{-\frac{t}{k}} f(t) \ \mathrm{d}t$$

Where, k has unit of time.

Shehu Transform:
Shehu Maitama and Weidong zhao in 2019 introduced Shehu transform which generalizes both the Laplace transform, which is applied to both ordinary and partial differential equations.
Shehu transform of function $f(t)$ is defined as,

$$\mathbb{S}\{f(t)\} = \int_0^{\infty} e^{\frac{-st}{u}} f(t)\, \mathrm{d}t$$

ARA Transform:
The ARA integral Transform was introduced in 2020 by mathematician R. Saadeh and Ahmad Qazza. This transform is used to simplify and solve complex differential and integral equations used in engineering and physics.
The ARA transform of continuous function $f(t)$ defined as,

$$\mathscr{G}\{f(t)\}(s) = s \int_0^{\infty} e^{-st} f(t)\, \mathrm{d}t \quad , \quad s > 0$$

Sawi Transform:
A new transform called Sawi transform was introduced in 2019 by mathematician Mohand Mahgoub. The transform is defined for $f(t)$ as follows,

$$W\{f(t)\} = \frac{1}{v^2} \int_0^{\infty} f(t)\, \mathrm{d}t$$

Double integral Transform:
In this section we discuss double integral transform of all above-mentioned integral transforms and their basic properties and differential properties.

Fourier-Laplace Transform:
Fourier-Laplace Transform for the function $f(t,x)$ is defined as

$$FL\{f(t,x)\} = F(s,p) = \int_{-\infty}^{\infty} \int_0^{\infty} e^{-i(st - ipx)}\, f(t,x)\, \mathrm{d}t\,\mathrm{d}x$$

Fourier-Laplace Inverse is given by

$$F^{-1}L-1\{F(s,p)\} = f(t,x) = \lim_{r,\tau \to \infty} \frac{1}{4\pi^2} \int_{-r}^{r} \int_{\tau}^{\tau} e^{-i(st - ipx)}\, F(s,p)\, \mathrm{d}s\,\mathrm{d}p\ .$$

Fourier-Laplace transforms of some basic functions:

Sr. No.	$f(t,x)$	$FL\{f(t,x)\} = F(s,p)$
1	**1**	$\frac{-i}{sp}$
2	$t\ x$	$\frac{-1}{s^2\ p^2}$
3	$t^n\ x^n$	$\frac{(-1)^{-n-1}\,(i)^{-n-1}\,(n!)^2}{s^{n+1}\ p^{n+1}}$
4	e^{at+bx}	$\frac{1}{(is-a)\,(p-b)}$
5	$\mathrm{Sin}at\ \ \sin bx$	$\frac{ab}{(a^2-s^2)\,(b^2-p^2)}$
6	$\mathrm{Cos}at\ \ \cos bx$	$\frac{isp}{(a^2-s^2)\,(b^2-p^2)}$

7	Sinhat sinhbx	$\frac{-ab}{(s^2+a^2)(p^2-b^2)}$
8	Coshat coshbx	$\frac{-isp}{(s^2+a^2)(p^2-b^2)}$

Differential Property:

1. $FLf_x(t,x) = pFLf(t,x) - k$
2. $FLf_{xx}(t,x) = p^2FLf(t,x) - pk$
3. $FLf^n_x(t,x) = p^n\, FLf(t,x) - p^{n-1}\, k$

Where
$$k = \int_{-\infty}^{\infty} e^{-ist\, f(t,0)}\, \mathrm{d}t$$

First Shifting theorem:

$$FL\{e^{-ax-bt} f(t,x)\} = F(s-ib\ , p+a)$$

Laplace-ARA Transform:

Let $u(x,t)$have a continuous function of two positive variables $x\,\&\,t$. Then Laplace-ARA transform of $u(x,t)$is defined as

$$L_x\, \mathcal{G}_t\{u(x,t)\} = Q(v,s) = s\int_0^{\infty}\int_0^{\infty} e^{-(vx+st)}\, u(x,t)\mathrm{d}x\mathrm{d}t \ , \qquad v,s>0$$

The inverse of the Laplace -ARA Transform is found using:

$$L_x^{-1}\, \mathcal{G}_t^{-1}\{Q(v,s)\} = u(x,t) = \frac{1}{(2\pi i)}\int_{c-i\infty}^{c+i\infty} e^{vx}\, dv\ \ \frac{1}{(2\pi i)}\int_{r-i\infty}^{r+i\infty} \frac{e^{st}}{s}\, Q(v,s)\mathrm{d}s\ .$$

Laplace-ARA Transform of some basic functions:

Sr. No.	$u(x,t)$	$L_x\, \mathcal{G}_t\{u(x,t)\} = Q(v,s)$
1	1	$\frac{1}{v}$
2	$x^{\alpha}\, t^{\beta}$	$\frac{\alpha!\ \beta!}{v^{\alpha+1}\, s^{\beta}}$, $\alpha,\beta > -1$
3	e^{ax+bt}	$\frac{s}{(v-\alpha)(s-\beta)}$
4	$\mathrm{Sin}(\alpha x+\beta t)$	$\frac{s\ (v\beta+s\alpha)}{(v^2+\alpha^2)(s^2+\beta^2)}$
5	$\mathrm{Cos}(\alpha x+\beta t)$	$\frac{s\ (sv-\alpha\beta)}{(v^2+\alpha^2)(s^2+\beta^2)}$
6	$\mathrm{Sin}h(\alpha x+\beta t)$	$\frac{s\ (v\beta+s\alpha)}{(v^2-\alpha^2)(s^2-\beta^2)}$
7	$\mathrm{Cos}h(\alpha x+\beta t)$	$\frac{s\ (sv+\alpha\beta)}{(v^2-\alpha^2)(s^2-\beta^2)}$

Differential Property:

1. $L_x \mathscr{G}_t\{u_t(x,t)\} = s\,Q(v,s) - s\,L\{u(x,0)\}$
2. $L_x \mathscr{G}_t\left\{u_x(x,t)\right\} = v\,Q(v,s) - \mathscr{G}\{u(0,t)\}$
3. $L_x \mathscr{G}_t\left\{u_{tt}(x,t)\right\} = s^2\,Q(v,s) - s^2\,L\{u(x,0)\} - s\,L\left\{u_t(x,0)\right\}$
4. $L_x \mathscr{G}_t\left\{u_{xx}(x,t)\right\} = v^2\,Q(v,s) - v\,\mathscr{G}\{u(0,t)\} - \mathscr{G}\left\{u_x(0,t)\right\}$
5. $L_x \mathscr{G}_t\left\{u_{xt}(x,t)\right\} = vs\,Q(v,s) - s\,\mathscr{G}\{u(0,t)\} - vs\,L\{u(x,0)\} + s\,u(0,0)$

First Shifting theorem:

$$L_x \mathscr{G}_t\{e^{\alpha x+\beta t}\,u(x,t)\} = \frac{s}{(s-\beta)}Q(v-\alpha,\, s-\beta)$$

Sumudu-Sawi Transform:

The Sumudu-Sawi transform of function $h(\xi,x)$ defined as,

$$S_\xi W_x\{h(\xi,x)\} = H(\delta,\varepsilon) = \frac{1}{\delta\,\varepsilon^2}\int_0^\infty\int_0^\infty e^{-\left(\frac{\xi}{\delta}-\frac{x}{\varepsilon}\right)} h(\xi,x)\,d\xi\,dx\ .$$

Sumudu-Sawi Transform of some basic functions:

Sr. No.	$h(\xi,x)$	$S_\xi W_x\{h(\xi,x)\} = H(\delta,\varepsilon)$
1	1	$\frac{1}{\varepsilon}$
2	$\xi^\mu\, x^v$	$\delta^\mu\,\varepsilon^{v-1}\ \ \mu!\ \ v!\qquad Re\left(\frac{1}{\delta}\right)>0\,,\quad Re(\mu)>-1$
3	$e^{\mu\xi+vx}$	$\frac{1}{\varepsilon\,(1-\delta\mu)\,(1-v\varepsilon)},\quad Re\left(\frac{1}{\delta}\right) > Re(\mu)$
4	$e^{i(\mu\xi+vx)}$	$\frac{-1}{\varepsilon\,(i+\delta\mu)\,(i+v\varepsilon)},\quad Re\left(\frac{1}{\delta}\right)+Img(\mu)>0$
5	$Sin(\mu\xi+vx)$	$\frac{(\mu\delta+\varepsilon v)}{\varepsilon\left(1+\mu^2\delta^2\right)\left(1+v^2\varepsilon^2\right)}$ $\lvert Img(\mu)\rvert < Re\left(\frac{1}{\delta}\right)$
6	$Cos(\mu\xi+vx)$	$\frac{(1-\mu\delta\varepsilon v)}{\varepsilon\left(1+\mu^2\delta^2\right)\left(1+v^2\varepsilon^2\right)}$ $\lvert Img(\mu)\rvert < Re\left(\frac{1}{\delta}\right)$
7	$Sinh(\mu\xi+vx)$	$\frac{(\mu\delta+\varepsilon v)}{\varepsilon\left(1-\mu^2\delta^2\right)\left(1-v^2\varepsilon^2\right)}$ $Re\left(\frac{1}{\delta}\right) > Re(\mu)\ \ \&\ \ Re\left(\frac{1}{\delta}+\mu\right)>0$
8	$Cosh(\mu\xi+vx)$	$\frac{(1+\mu\delta\varepsilon v)}{\varepsilon\left(1-\mu^2\delta^2\right)\left(1-v^2\varepsilon^2\right)}$ $Re(\mu) < Re\left(\frac{1}{\delta}\right)\ \ \&\ \ Re\left(\frac{1}{\delta}+\mu\right)>0$

Differential Property:

1. $S_{\xi} W_{x}\left\{h_{\xi}(\xi,x)\right\} = \frac{1}{\delta} H(\delta,\varepsilon) - \frac{1}{\delta} W\{h(0,x)\}$
2. $S_{\xi} W_{x}\left\{h_{\xi\xi}(\xi,x)\right\} = \frac{1}{\delta^2} H(\delta,\varepsilon) - \frac{1}{\delta^2} W\{h(0,x)\} - \frac{1}{\delta} W\left\{h_{\xi}(0,x)\right\}$
3. $S_{\xi} W_{x}\left\{h_{x}(\xi,x)\right\} = \frac{1}{\varepsilon} H(\delta,\varepsilon) - \frac{1}{\varepsilon^2} S\{h(\xi,0)\}$
4. $S_{\xi} W_{x}\left\{h_{xx}(\xi,x)\right\} = \frac{1}{\varepsilon^2} H(\delta,\varepsilon) - \frac{1}{\varepsilon^2} S\{h(\xi,0)\} - \frac{1}{\varepsilon^2} S\left\{h_{x}(\xi,0)\right\}$
5. $S_{\xi} W_{x}\left\{h_{\xi x}(\xi,x)\right\} = \frac{1}{\varepsilon\delta} H(\delta,\varepsilon) - \frac{1}{\delta\varepsilon^2} S\{h(\xi,0)\} - \frac{1}{\delta\varepsilon} W\{h(0,x)\} + \frac{1}{\delta\varepsilon^2} h(0,0)$

Sumudu-Shehu Transform:

The Sumudu-Shehu transform of function $q(r,v)$ defined as,

$$S_r H_v\{q(r,v)\} = Q(k,\lambda,\mu) = \frac{1}{k}\int_0^\infty \int_0^\infty e^{-\left(\frac{r}{k} - \frac{\lambda}{\mu} v\right)} q(r,v)\,\mathrm{d}r\ \mathrm{d}v \ .$$

Sumudu-Shehu Transform of some basic functions:

Sr. No.	$q(r,v)$	$S_r H_v\{q(r,v)\} = Q(k,\lambda,\mu)$
1	1	$\frac{\mu}{\lambda}$
2	$r^{\beta}\ v^{\tau}$	$\frac{k^{\beta}\ \mu^{\tau+1}\ \ \beta!\ \tau!}{\lambda^{\tau+1}}$ $Re(k) > 0$, $Re(\beta) > -1$
3	$e^{\beta r + \tau v}$	$\frac{\mu}{(1-k\beta)\ (\lambda - \tau\mu)}$, $Re\left(\frac{1}{k}\right) > Re(\beta)$
4	$e^{i(\beta r + \tau v)}$	$\frac{i\mu}{(i-k\beta)\ (\lambda - i\tau\mu)}$, $Re\left(\frac{1}{k}\right) + Img(\beta) > 0$
5	$Sin(\beta r + \tau v)$	$\frac{\mu(k\lambda\beta + \mu\tau)}{\varepsilon\left(1+k^2\beta^2\right)\left(\lambda^2 + \tau^2\mu^2\right)}$ $\lvert Img(\beta)\rvert < Re\left(\frac{1}{k}\right)$
6	$Cos(\beta r + \tau v)$	$\frac{\mu(\lambda - k\mu\beta\tau)}{\left(1+k^2\beta^2\right)\left(\lambda^2 + \tau^2\mu^2\right)}$ $\lvert Img(\beta)\rvert < Re\left(\frac{1}{k}\right)$
7	$Sinh(\beta r + \tau v)$	$\frac{\mu(k\lambda\beta + \mu\tau)}{\varepsilon\left(k^2\beta^2 - 1\right)\left(\lambda^2 - \tau^2\mu^2\right)}$ $Re\left(\frac{1}{k}\right) > Re(\beta)$ & $Re\left(\frac{1}{k} + \beta\right) > 0$
8	$Cosh(\beta r + \tau v)$	$\frac{\mu(\lambda + k\mu\beta\tau)}{\left(k^2\beta^2 - 1\right)\left(\lambda^2 - \tau^2\mu^2\right)}$ $Re\left(\frac{1}{k}\right) > Re(\beta)$ & $Re\left(\frac{1}{k} + \beta\right) > 0$

Differential Property:

1. $S_r H_v\left\{q_r(r,v)\right\} = \frac{1}{k}Q(k,\lambda,\mu) - \frac{1}{k}H\{q(0,v)\}$
2. $S_r H_v\left\{q_{rr}(r,v)\right\} = \frac{1}{k^2}Q(k,\lambda,\mu) - \frac{1}{k^2}H\{q(0,v)\} - \frac{1}{k}H\left\{q_r(0,v)\right\}$
3. $S_r H_v\left\{q_v(r,v)\right\} = \frac{\lambda}{\mu}Q(k,\lambda,\mu) - S\{q(r,0)\}$
4. $S_r H_v\left\{q_{vv}(r,v)\right\} = \frac{\lambda^2}{\mu^2}Q(k,\lambda,\mu) - \frac{\lambda}{\mu}S\{q(r,0)\} - S\left\{q_v(r,0)\right\}$
5. $S_r H_v\left\{q_{rv}(r,v)\right\} = \frac{\lambda}{k\mu}Q(k,\lambda,\mu) - \frac{1}{k}S\{q(r,0)\} - \frac{\lambda}{k\mu}H\{q(0,v)\} + \frac{1}{k}q(0,0)$

Laplace-Sawi Transform:

The Laplace-Sawi Transform of function $g(\eta,\theta)$ defined as,

$$L_\eta W_\theta\{g(\eta,\theta)\} = G(\delta,\varepsilon) = \frac{1}{\varepsilon^2}\int_0^\infty\int_0^\infty e^{-\left(\delta\eta - \frac{\theta}{\varepsilon}\right)} g(\eta,\theta)\,d\eta\, d\theta \;,$$

Where $g(\eta,\theta)$ is continuous function on $(0,\infty) \times (0,\infty)$.

Laplace- Sawi Transform of some basic functions:

Sr. No.	$g(\eta,\theta)$	$L_\eta W_\theta\{g(\eta,\theta)\} = G(\delta,\varepsilon)$
1	**1**	$\frac{1}{\delta\varepsilon}$
2	$\eta^\mu\ \theta^\nu$	$\frac{\varepsilon^{\nu-1}\ \mu!\ \nu!}{\delta^{\mu+1}}$, $Re(\delta) > 0$, $Re(\mu) > -1$
3	$e^{\mu\eta+\nu\theta}$	$\frac{1}{\varepsilon\,(\delta-\mu)\,(1-\nu\varepsilon)}$, $Re(\delta) > Re(\mu)$
4	$e^{i(\mu\eta+\nu\theta)}$	$\frac{i}{\varepsilon\,(\delta-i\mu)\,(i+\nu\varepsilon)}$, $Re(\delta) + Img(\mu) > 0$
5	$Sin(\mu\eta+\nu\theta)$	$\frac{(\mu+\delta\varepsilon\nu)}{\varepsilon(\delta^2+\mu^2)(1+\nu^2\varepsilon^2)}$ $Img(\mu) < Re(\delta)$
6	$Cos(\mu\eta+\nu\theta)$	$\frac{(\delta+\varepsilon\mu\nu)}{\varepsilon(\delta^2+\mu^2)(1+\nu^2\varepsilon^2)}$ $Img(\mu) < Re(\delta)$
7	$Sinh(\mu\eta+\nu\theta)$	$\frac{(\mu+\delta\varepsilon\nu)}{\varepsilon(\delta^2-\mu^2)(1-\nu^2\varepsilon^2)}$ $Re(\mu) < Re(\delta)$ and $Re(\mu) + Re(\delta) > 0$
8	$Cosh(\mu\eta+\nu\theta)$	$\frac{(\delta+\varepsilon\mu\nu)}{\varepsilon(\delta^2-\mu^2)(1-\nu^2\varepsilon^2)}$

Differential Property:

1. $L_\eta W_\theta\left\{g_\eta(\eta,\theta)\right\} = \delta\, G(\delta,\varepsilon) - W\{g(0,\theta)\}$
2. $L_\eta W_\theta\left\{g_{\eta\eta}(\eta,\theta)\right\} = \delta^2\, G(\delta,\varepsilon) - \delta\, W\{g(0,\theta)\} - W\left\{g_n(0,\theta)\right\}$
3. $L_\eta W_\theta\left\{g_\theta(\eta,\theta)\right\} = \frac{1}{\varepsilon}\, G(\delta,\varepsilon) - \frac{1}{\varepsilon^2}\, L\{g(\eta,0)\}$
4. $L_\eta W_\theta\left\{g_{\theta\theta}(\eta,\theta)\right\} = \frac{1}{\varepsilon^2}\, G(\delta,\varepsilon) - \frac{1}{\varepsilon^3}\, L\{g(\eta,0)\} - \frac{1}{\varepsilon^2}\, L\left\{g_\theta(\eta,0)\right\}$
5. $L_\eta W_\theta\left\{g_{\eta\theta}(\eta,\theta)\right\} = \frac{\delta}{\varepsilon}\, G(\delta,\varepsilon) - \frac{\delta}{\varepsilon^2}\, L\{g(\eta,0)\} - \frac{1}{\varepsilon}\, W\{g(0,\theta)\} + \frac{1}{\varepsilon^2}\, g(0,\theta)$

ARA - Sawi Transform:

The ARA-Sawi Transform of function $g(\varrho,\sigma)$ defined as,

$$A_\varrho\, W_\sigma\{g(\varrho,\sigma)\} = G(\lambda,w) = \frac{\lambda}{w^2}\int_0^\infty\int_0^\infty e^{\left(-\lambda\varrho - \frac{\sigma}{w}\right)}\, g(\varrho,\sigma)\,d\varrho\, d\sigma \;,$$

Where $g(\varrho,\sigma)$ is continuous function on $(0,\infty) \times (0,\infty)$.
The inverse ARA-Sawi transform is given by,

$$A_\varrho^{-1}\, W_\sigma^{-1}\{G(\lambda,w)\} = g(\varrho,\sigma) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{\lambda\varrho}}{\lambda}\,d\lambda\;\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{e^{\sigma/w}}{w^2}\,G(\lambda,w)\,dw.$$

ARA-Sawi Transform of some basic functions:

Sr. No.	$g(\varrho,\sigma)$	$A_\varrho\, W_\sigma\{g(\varrho,\sigma)\} = G(\lambda,w)$
1	1	$\frac{1}{w}\,,\; Re(\lambda) > 0$
2	$\varrho^r\,\sigma^\delta$	$\frac{w^{\delta-1}\; r!\;\delta!}{\lambda^r}\;,\quad Re(\lambda) > 0\,,\quad Re(r) > -1$
3	$e^{r\varrho + \delta\sigma}$	$\frac{\lambda}{w\,(\lambda - r)\,(1-\delta w)},\quad Re(\lambda) > Re(r)$
4	$e^{i(r\varrho + \delta\sigma)}$	$\frac{i\lambda}{w\,(\lambda - ir)\,(i-\delta w)},\quad Re(\lambda) + Img(r) > 0$
5	$Sin(r\varrho + \delta\sigma)$	$\frac{\lambda(r + \lambda w\delta)}{w\left(\lambda^2 + r^2\right)\left(1 + \delta^2 w^2\right)}$ $Img(r) < Re(\lambda)$
6	$Cos(r\varrho + \delta\sigma)$	$\frac{\lambda(\lambda - rw\delta)}{w\left(\lambda^2 + r^2\right)\left(1 + \delta^2 w^2\right)}$ $\lvert Img(r)\rvert < Re(\lambda)$
7	$Sinh(r\varrho + \delta\sigma)$	$\frac{\lambda(r + \lambda w\delta)}{w\left(\lambda^2 - r^2\right)\left(1 - \delta^2 w^2\right)}$ $Re(r) < Re(\lambda)\;\; and\;\; Re(r) + Re(\lambda) > 0$

8	$Cosh(r\varrho+\delta\sigma)$	$\frac{\lambda(\lambda+r\,w\delta)}{w(\lambda^2-r^2)(1-\delta^2w^2)}$ $Re(r)< Re(\lambda)$ and $Re(r)+ Re(\lambda)>0$

Differential Property:

1. $A_{\varrho} W_{\sigma}\left\{g_{\varrho}(\varrho,\sigma)\right\} = \lambda\, G(\lambda,w) - \lambda\, W\{g(0,\sigma)\}$
2. $A_{\varrho} W_{\sigma}\left\{g_{\varrho\varrho}(\varrho,\sigma)\right\} = \lambda^2 G(\lambda,w) - \lambda^2 W\{g(0,\sigma)\} - \lambda\, W\left\{g_{\varrho}(0,\sigma)\right\}$
3. $A_{\varrho} W_{\sigma}\left\{g_{\sigma}(\varrho,\sigma)\right\} = \frac{1}{w} G(\lambda,w) - \frac{1}{w^2} A\{g(\varrho,0)\}$
4. $A_{\varrho} W_{\sigma}\left\{g_{\sigma\sigma}(\varrho,\sigma)\right\} = \frac{1}{w^2} G(\lambda,w) - \frac{1}{w^3} A\{g(\varrho,0)\} - \frac{1}{w^3} A\left\{g_{\sigma}(\varrho,0)\right\}$
5. $A_{\varrho} W_{\sigma}\left\{g_{\varrho\sigma}(\varrho,\sigma)\right\} = \frac{\lambda}{w} G(\lambda,w) - \frac{\lambda}{w^2} A\{g(\varrho,0)\} - \frac{\lambda}{w} W\{g(0,\sigma)\} + \frac{\lambda}{w^2} g(0,0)$

ARA-Sumudu Transform:

The ARA-Sumudu Transform of function $g(x,y)$ defined as,

$$\mathcal{G}_x S_y\{g(x,y)\} = G(s,\mu) = \frac{1}{\mu}\int_0^\infty \int_0^\infty e^{\left(-sx-\frac{y}{\mu}\right)} g(x,y)\mathrm{d}x\ \mathrm{d}y \quad , \quad s>0\ ,\ \mu>0$$

The inverse ARA-Sumudu is given by

$$\mathcal{G}_x^{-1} S_y^{-1}\{G(s,\mu)\} = g(x,y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{sx}}{s} ds \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty} \frac{e^{\frac{y}{\mu}}}{\mu} G(s,\mu)\ \mathrm{d}\mu \ .$$

ARA-Sumudu Transform of some basic functions:

Sr. No.	$g(x,y)$	$\mathcal{G}x\, S_y\{g(x,y)\} = G(s,\mu)$
1	1	1
2	$x^a\ y^b$	$\frac{\mu^b\ a!\ b!}{s^a}$
3	e^{ax+by}	$\frac{s}{(s-a)(1-b\mu)}$
4	$e^{i(ax+by)}$	$\frac{is}{(s-ia)(b\mu+1)}$
5	$Sin(ax+by)$	$\frac{s(a+bs\mu)}{(a^2+s^2)(b^2\mu^2+1)}$
6	$Cos(ax+by)$	$\frac{s(s-ba\mu)}{(a^2+s^2)(b^2\mu^2+1)}$
7	$Sinh(ax+by)$	$\frac{s(a+bs\mu)}{(a^2-s^2)(b^2\mu^2-1)}$
8	$Cosh(ax+by)$	$\frac{s(s+ba\mu)}{(a^2-s^2)(b^2\mu^2-1)}$

Differential Property:

1. $\mathscr{G}_x S_y\left\{g_x(x,y)\right\} = s\, G(s,\mu) - s\, S\{g(0,y)\}$
2. $\mathscr{G}_x S_y\left\{g_y(x,y)\right\} = \frac{1}{\mu} G(s,\mu) - \frac{1}{\mu}\mathscr{G}\{g(x,0)\}$ 3. $\mathscr{G}_x S_y\left\{g_{xx}(x,y)\right\} = s^2 G(s,\mu) - s^2 S\{g(0,y)\} - s\, S\left\{g_x(0,y)\right\}$
4. $\mathscr{G}_x S_y\left\{g_{yy}(x,y)\right\} = \frac{1}{\mu^2} G(s,\mu) - \frac{1}{\mu^2}\mathscr{G}\{g(x,0)\} - \frac{1}{\mu}\mathscr{G}\left\{g_y(x,0)\right\}$
5. $\mathscr{G}_x S_y\left\{g_{xy}(x,y)\right\} = \frac{s}{\mu} G(s,\mu) - S\{g(0,y)\} - \mathscr{G}\{g(x,0)\} - g(0,0)$

Key Theoretical and Behavioral Characteristics:

The Fourier-Laplace Transform:
Non symmetric kernel $e^{-i(st-ipx)}$ seamlessly splits steady spatial wave oscillations from decaying temporal dissipation.

The Laplace-ARA Transform:

Laplace ARA Transform combines classic exponential dampening with the ARA kernel se^{-st}. This combination offers superior boundary localization for singular partial derivatives.

The Sumudu-Sawi Transform:

Unit preserving scaling transform via dual non exponential kernels. It isolates parameter values directly without distorting the functions structural dimensions.

The Laplace-Sawi Transform:

Hybrid design leveraging Sawi's unique $\frac{1}{\varepsilon^2}e^{-\theta/\varepsilon}$ kernel. This significantly speeds up the conversation of volatile second order rates of change.

The ARA-Sawi Transform:

Highly focused algebraic simplification tool. It avoids complex integration by parts terms typically generated by standard Laplace method.

The ARA-Sumudu Transform:
Combines the powerful derivative handling feature of the ARA transform with the smooth, dimensionless scaling of the Sumudu-Transform.

The Sumudu-Shehu Transform:
Uses a variable dependent Shehu Kernel $e^{-\frac{s}{u}t}$ to handle coupled systems of fractional equations that have non-local boundaries.

Critical Comparison of Combined Integral Transforms:

Transform	Major Strengths	Limitations	Applications
Fourier–Laplace Transform	Combines the spatial analysis capability of the Fourier transform with the temporal analysis of the Laplace transform. It effectively converts PDEs involving both space and time into algebraic equations and naturally accommodates initial conditions.	Requires existence conditions for both Fourier and Laplace transforms simultaneously. Inverse transformation may become mathematically involved for multidimensional problems. Less suitable for nonlinear differential equations without additional analytical techniques.	Heat conduction, wave propagation, electromagnetics, diffusion, Quantum mechanics, Signal analysis. Electrodynamics and Structural Mechanics, Mathematical Biology
Laplace–ARA Transform	Simplifies the solution of linear ordinary and partial differential equations by combining the operational properties of the Laplace and ARA transforms. Efficient handling of initial-value problems with relatively simple inverse operations.	Mathematical theory is still developing compared with classical transforms. Limited availability of transform tables, software implementation, and benchmark applications. Few rigorous studies on convergence and uniqueness.	Initial-value problems, Mathematical physics, engineering models, reaction-diffusion equations. Partial differential equations like hyperbolic Klein-Gordon equation, structural wave equations and multivariable telegraph models
Laplace–Sawi Transform	Possesses strong differential properties inherited from the Laplace transform while exploiting the computational flexibility of the Sawi transform. Often reduces lengthy algebraic manipulations.	Limited theoretical development. Applications remain largely restricted to linear differential equations. Few studies investigate numerical stability and multidimensional extensions.	Ordinary differential equations, integral equations, Engineering mathematics, Mathematical Physics.
Sumudu–Sawi Transform	Preserves dimensional consistency through the Sumudu transform while improving computational simplicity using the Sawi transform. Frequently avoids variable scaling during transformation.	Less mature theoretical foundation. Limited literature addressing inverse transform algorithms and nonlinear systems. Computational performance has not been extensively validated for complex models.	Control systems, Voltera-integro partial differential equation, engineering analysis, transport equations, mathematical modelling.
Sumudu–ARA Transform	Combines the unit-preserving property of the Sumudu transform with the operational simplicity of the ARA transform. Suitable for solving engineering models where preservation of physical units is advantageous.	Relatively new transform with limited applications reported in literature. Lack of comprehensive convergence theory and numerical comparison with established transforms.	Heat equations, Wave equations, Advection-diffusion, equation Electrical engineering, Control Theory, Mathematical Biology, linear PDEs.

Sumudu–Shehu Transform	Integrates the advantages of the Sumudu and Shehu transforms, offering efficient treatment of initial-value problems and simplified inversion for several classes of equations. Applicable to fractional models in recent studies.	Limited standardization of notation across publications. Few comprehensive studies compare its computational efficiency with classical transforms. Extension to nonlinear multidimensional systems remains challenging.	Heat equation, Fractional differential equations, Engineering systems, fluid flow, Mathematical physics, Quantum mechanics wave mechanics.
Sawi–ARA Transform	Hybrid transforms providing simplified operational rules and efficient handling of differential equations. Useful in reducing computational complexity for certain linear models.	Very recent development with comparatively limited theoretical analysis. Inverse transform techniques, convergence criteria, and numerical implementations require further investigation. Fewer real-world applications have been reported.	Analytical solutions of ODEs, PDEs, Engineering Mathematics, Applied sciences.

Conclusion:-
This paper presents a comprehensive comparative analysis of several combined integral transforms, namely the Combined Fourier–Laplace Transform, Combined Laplace–ARA Transform, Combined Laplace–Sawi Transform, Combined Sumudu–Sawi Transform, Combined Sumudu–ARA Transform, Combined Sumudu–Shehu Transform, and Combined Sawi–ARA Transform. The study systematically examines their transform formulations for elementary functions, operational and differential properties, and underlying theoretical characteristics within a unified analytical framework. A comparative investigation of their mathematical behaviour reveals both common operational structures and distinct features that influence their applicability to different classes of mathematical problems. The analysis further demonstrates that these combined integral transforms possess significant computational advantages, including simplified treatment of initial and boundary conditions, efficient transformation of differential operators, and enhanced capability for obtaining analytical or semi-analytical solutions of ordinary and partial differential equations. Their broad applicability in mathematical physics, engineering and applied sciences underscores their importance as effective mathematical tools for modelling complex physical phenomena.
The principal contribution of this work lies in providing a unified comparative framework that facilitates the understanding of the operational characteristics, relative advantages, and application potential of contemporary combined integral transforms. The results presented herein may serve as a useful reference for researchers in selecting appropriate transform techniques for specific mathematical models and engineering applications.

Future research may focus on extending these comparative investigations to newly developed hybrid integral transforms, establishing additional operational theorems, and exploring their applications to fractional differential equations, fractional partial differential equations, stochastic models, nonlinear dynamical systems, integral equations, and coupled multi-physics problems. Furthermore, the development of numerical algorithms and computational implementations based on these combined transforms offers promising opportunities for solving large-scale and real-world scientific and engineering problems with greater accuracy and computational efficiency.

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