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### RESEARCH ARTICLE

## OPTIMIZATION OF MIXED VEHICLE-DRONE ROUTING WITH TIME WINDOWS FOR LAST-MILE DELIVERY

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### Abstract

The growth of e-commerce and the demand for ever shorter delivery lead times put considerable pressure on last-mile logistics operators. This article proposes a unified mixed-integer linear programming (MILP) model for mixed vehicle-drone routing with time windows (VRPTW-D). A conventional vehicle may, at each of its stops, launch a drone that serves a customer on its own and is then recovered by the vehicle, either at the same stop or at a later stop. The compact formulation relies on ternary flight variables, continuous time variables and an exact linearization of the vehicle-drone synchronization; it embeds valid inequalities (Dantzig – Fulkerson - Johnson subtour elimination and use - flow linking) that strengthen the continuous relaxation. For larger instances, we develop an ALNS (Adaptive Large Neighborhood Search) heuristic calibrated against the exact solutions. The exact model proves optimality on small instances ( $n \leq 10$ ) and, beyond, provides feasible solutions together with bounds; the ALNS is the practical solution method for large instances. A reproducible instance set, generated in the style of Solomon (classes C1, R1 and RC1) with an extension of the time windows to drones, allows us to measure precisely the contribution of the drones, the value of the relaxation and the comparative performance of the exact model and the ALNS, complemented by a sensitivity analysis (drone endurance, relative speed, time-window width).

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### Introduction:-

Last-mile delivery designates the final part of the supply chain, the one that connects the last urban warehouse or consolidation point to the end customers. It concentrates a disproportionate share of total distribution costs: according to recent studies, it accounts for between 20% and 50% of total logistics costs and for a large part of urban greenhouse-gas emissions (Boysen, Fedtke, and Schwerdfeger, 2021; Chiang, Li, Shang, and Urban, 2019). The growth of e-commerce, the multiplication of delivery time slots offered to consumers and urban congestion make this segment increasingly costly and complex to organize. Civil aerial drones have been, for a decade, a major

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candidate technology to transform this activity. Their main advantage is kinematic, a drone flies in a straight line at a speed higher than that of a road vehicle, unconstrained by the road network or by congestion. Their main disadvantage is operational: low payload (one parcel), limited endurance (a few tens of kilometers), regulatory and meteorological constraints, and battery maintenance costs.

Between the exclusive use of vehicles and the exclusive use of drones, the scientific literature has quickly identified a particularly promising hybrid solution: a mothership vehicle carries one or several drones and operates them in a coordinated way. The vehicle travels on the road network; at certain stops, it launches a drone that performs a direct flight to a customer and is then recovered, either at the same stop or at a later stop. This principle, introduced under the name Flying Sidekick Traveling Salesman Problem (FSTSP) by Murray and Chu (2015), reduces simultaneously the distance traveled by the vehicle and the overall tour time, while keeping the flexibility of road delivery. Most of the ensuing works deal with the problem without time constraints or with time windows restricted to the vehicles. Yet in last-mile practice, the promised delivery time is a central contractual element: each customer has a time window  $[a_i, b_i]$  during which delivery must be performed.

**The objective of this article is to fill this gap by proposing a model and a solution method for the vehicle routing problem with drones and time windows (VRPTW-D), in which:-**

- each customer must be served exactly once, either by a vehicle or by a drone;
- each drone is launched from a vehicle (or from the depot) and is recovered by the same vehicle (or at the depot), with a limited flight endurance;
- the time windows are soft: lateness is allowed but penalized, which reflects the operational trade-off between meeting commitments and the tour cost;
- the objective minimizes the total cost: fixed vehicle-usage costs, travel costs, drone flight energy cost and lateness penalties.

**The contributions of this article are the following:-**

1. A unified MILP formulation of the VRPTW-D integrating simultaneously the vehicle tours, the selection of drone flights and the management of soft time windows, with an exact linearization of the temporal synchronization between the mothership vehicle and its drones.
2. A compact formulation: drone flights are represented by ternary variables  $w^d_{ijk}$  (launch at  $i$ , serve  $j$ , recover at  $k$ ), combined with flow variables  $x^v_{ij}$  for the vehicles, continuous time variables  $T^v_i$  and  $q^d_{ijk}$ , and penalty variables  $p_j$ . Subtours are eliminated by time progression, strengthened by Dantzig–Fulkerson–Johnson (DFJ) inequalities separated dynamically.
3. Valid inequalities and a lower bound computed by strengthened continuous relaxation, used to quantify the quality of the heuristic solutions.
4. An ALNS heuristic (Adaptive Large Neighborhood Search) for large instances, whose parameters are calibrated on the exact solutions of small instances.
5. A reproducible benchmark built on Solomon’s instance model (Solomon, 1987) (classes C1, R1, RC1), extended to drones (sets of candidate flights, endurance, drone time windows), together with a sensitivity analysis to the key operational parameters (endurance, relative speed, time-window width).

This positioning is explicit, the exact model serves as a validation reference on small instances (optimality proof) and as an incumbent generator on larger ones, while the ALNS is the operational solution method for real-size instances. The rest of the article is organized as follows. Section 2 presents the literature review. Section 3 formalizes the problem. Section 4 describes the MILP model and its strengthenings. Section 5 details the ALNS heuristic. Section 6 presents the experimental protocol and the results. Section 7 concludes and offers perspectives.

## Literature Review:-

### From Routing Problems to Drone Delivery:-

The vehicle routing problem with time windows (VRPTW) is one of the most studied combinatorial optimization problems since the foundational work of Solomon (1987). It generalizes the vehicle routing problem (VRP) by requiring that the visit of each customer falls within a time interval. The exact methods (dynamic programming, column generation, branch-and-cut) and heuristic methods (local search, metaheuristics) applicable are surveyed in Toth and Vigo (2014) and Cordeau, Desaulniers, Desrosiers, Solomon, and Soumis (2002). The introduction of aerial drones gave rise to a family of drone-assisted delivery problems. Murray and Chu (2015) formalize the FSTSP: a single vehicle, a single drone, the vehicle can launch the drone at one point of the tour and recover it

further. Agatz, Bouman, and Schmidt (2018) study the traveling salesman problem with drone (TSP-D) and propose several formulations as well as bounds. Ha, Deville, Pham, and Hà (2018) consider a minimum-cost variant and Wang, Poikonen, and Golden (2017) establish worst-case results for the VRP with drones. Poikonen, Wang, and Golden (2017) extend these models and connect them to the existing literature. Otto, Agatz, Campbell, Golden, and Pesch (2018) and Khoufi, Laouiti, and Adjih (2019) provide exhaustive surveys of the UAV variants of the TSP and the VRP.

#### **Drone-Assisted Delivery and Time Windows:-**

The literature on time windows in vehicle-drone problems (VRPTW-D) remained small until recently but is now expanding quickly. Three families of contributions can be distinguished. Compact formulations solved by general-purpose solvers. Di Puglia Pugliese and Guerriero (2017) model the delivery process of a fleet of trucks equipped with drones as a variant of the VRPTW in which a drone can be launched several times along a truck route, serve one customer per flight and be recovered at a later point, under a maximum waiting time for the truck; a striking conclusion of their study is that, under strict time windows and classical cost accounting, drones are not economically beneficial. Di Puglia Pugliese, Macrina, and Guerriero (2021) revisit the problem with a mixed-integer linear program (MILP) solved up to  $n = 15$  customers and a two-phase multi-start heuristic, on instances derived from VRPTW benchmarks. Coindreau, Gallay, and Zufferey (2021) propose a MILP complemented by an ALNS for a realistic cost structure (fixed daily fares, driver wages, fuel and electricity); on instances of up to 100 parcels they report savings up to 34% compared with truck-only delivery. Kuo, Lu, Lai, and Mara (2022) present a MIP model and a variable neighborhood search (VNS) heuristic for the VRPTW-D able to tackle up to 50 customers.

Decomposition-based exact methods. To push the provably optimal sizes further, recent works rely on column-generation schemes. Li and Wang (2023) design a branch-price-and-cut (BPC) for the truck-drone routing problem with time windows (TDRP-TW), where several drones can be carried by each truck and synchronized en route; their algorithm solves up to 50 customers to optimality and approximates instances of 100 customers with an ALNS. Yin, Li, Wang, Ignatius, Cheng, and Wang (2023) address the truck-based drone delivery routing problem with time windows (TDDRP), allowing a drone to serve several customers in a single flight, and solve it by branch-and-price-and-cut with a bidirectional labeling algorithm on instances of up to 45 customers. Schmidt (2024) develops the first exact algorithm for the VRP-DTW with the objective of minimizing the sum of route durations, via a BPC solving instances of up to 30 customers, while Sippel and Forbes (2024) propose a fragment-based exact approach for the TDDRP reaching instances of 125 customers. In the same period, Zhen, Gao, Tan, Wang, and Baldacci (2023) develop a branch-price-and-cut for a cooperative truck-drone delivery problem without explicit customer time windows.

Heuristics for realistic sizes and richer features. Beyond exact methods, Sacramento, Pisinger, and Ropke (2019) propose an ALNS for the vehicle routing problem with drones (VRPD), handling in particular time windows; it is one of the works closest to ours. Meng, Chen, and Li (2024a) solve a multi-visit drone-assisted pickup and delivery problem with time windows by valid inequalities and an ALNS. Meng, Li, Liu, and Chen (2024b) address a multi-visit drone-assisted routing problem with soft time windows and stochastic truck travel times within a two-stage stochastic model solved by a hybrid metaheuristic with sample average approximation. Two observations emerge. First, most of these works impose hard time windows and minimize a routing cost; the soft-window framework, which reflects the operational trade-off between service quality and tour cost, is addressed only by Meng et al. (2024b), in a stochastic setting. Second, the objective is almost always the total routing cost, except Schmidt (2024) who minimizes the route duration. Our contribution combines a soft-window formulation with an economic objective and a compact MILP, a combination absent from this literature.

#### **Qualitative Comparison with the VRPTW-D Literature:-**

Table 1 positions this work in the VRPTW-D literature according to the modeling choices (nature of the time windows, structure of the drone flights, objective function) and to the solution methods (exact capacity and largest size handled heuristically).

**Table 1: Qualitative comparison with the VRPTW-D literature. TW: nature of the time windows; Drone flight: number of served customers per flight and of sorties per tour; Objective: total cost (fixed, travel, energy, and lateness penalties as applicable), n is the number of customers; “-” indicates not reported or not applicable.**

| Reference                               | Approach                           | TW   |  | Drone flight                  | Objective                    | Exact n         | Heuristic n |
|-----------------------------------------|------------------------------------|------|--|-------------------------------|------------------------------|-----------------|-------------|
| Di Puglia Pugliese and Guerriero (2017) | Compact formulation                | hard |  | single-visit, several sorties | transport cost               | $\leq 9$        | -           |
| Di Puglia Pugliese et al. (2021)        | MILP + multi-start heuristic       | hard |  | single-visit, several sorties | total cost                   | 15              | larger      |
| Coindreau et al. (2021)                 | MILP + ALNS                        | hard |  | single-visit, several sorties | wages, fuel, energy          | -               | 100         |
| Kuo et al. (2022)                       | MIP + VNS                          | hard |  | single-visit                  | travel cost                  | -               | 50          |
| Li and Wang (2023)                      | BPC + ALNS (en-route sync.)        | hard |  | single-visit, several drones  | cost                         | 50              | 100         |
| Yin et al. (2023)                       | BPC                                | hard |  | multi-visit                   | cost                         | 45              | -           |
| Meng et al. (2024a)                     | MIBP + ALNS                        | hard |  | multi-visit                   | cost                         | -               | benchmark   |
| Meng et al. (2024b)                     | stochastic 2-stage + metaheuristic | soft |  | multi-visit                   | cost + penalties             | -               | benchmark   |
| Schmidt (2024)                          | BPC                                | hard |  | single-visit                  | route duration               | 30              | -           |
| Sippel and Forbes (2024)                | fragment approach                  | hard |  | multi-visit                   | routes, mileage              | 125*            | -           |
| This work                               | Compact MILP + valid ineq. + ALNS  | soft |  | single-visit, one sortie      | fixed, road, energy, penalty | 10 <sup>a</sup> | 100         |

\* Largest size reported by Sippel and Forbes (2024); some of these instances are solved approximately. <sup>a</sup> Optimality proven for all  $n = 10$  instances; for  $15 \leq n \leq 50$  the exact model provides a feasible solution together with a lower bound.

**Table 1 calls for four remarks:-**

- (i) Time windows. We are among the few works handling soft time windows (Meng et al., 2024b), but unlike Meng et al. (2024b) we do so in a deterministic framework with a compact MILP, which makes the model directly usable with any general-purpose solver.
- (ii) Solution method. The recent exact works rely on set-partitioning formulations solved by column generation (Li and Wang, 2023; Schmidt, 2024; Sippel and Forbes, 2024; Yin et al., 2023). These methods reach larger provably optimal instances (30-125 customers) but require a substantial algorithmic machinery. Our compact formulation is intentionally different: it is solved by CPLEX, provides a strengthened lower bound through valid inequalities exploited at the root, and matches the exact performance of the compact models of the literature on small instances ( $n \leq 15$ ). It therefore offers a simple, reproducible baseline against which the decomposition-based results of Li and

Wang (2023), Schmidt (2024), and Yin et al. (2023) can be compared, and our ALNS handles the same large sizes ( $n = 100$ ) as the ALNS-based works (Coindreau et al., 2021; Li and Wang, 2023).

(iii) Economic and managerial insights. Our economic objective combines the fixed fleet cost, the road cost, the drone energy cost and lateness penalties. With this structure, the drones generate a measured and reproducible gain of 15-49% (average 27.6% on the clustered class) over the drone-free solution, in line with the up-to-34% savings reported by Coindreau et al. (2021). This contrasts with the conclusion of Di Puglia Pugliese and Guerriero (2017) that drones are not economically justified under strict windows, the difference comes precisely from the penalty structure of the soft windows and from the fixed vehicle cost, which make the drone flights worth their energy consumption. Our sensitivity analysis (Section 6) further shows that the width of the time windows is the dominant cost lever, a point that reinforces the importance of modeling service-quality constraints in drone-assisted delivery.

(iv) Benchmark. Unlike most VRPTW-D works based on ad-hoc instances (Di Puglia Pugliese and Guerriero, 2017; Di Puglia Pugliese et al., 2021; Kuo et al., 2022) or on instances not made available (Li and Wang, 2023; Schmidt, 2024; Yin et al., 2023), our benchmark is fully reproducible (Solomon-style generation protocol, time windows extended to the drone flights, all parameter values reported), which directly addresses the reproducibility concern raised by the decentralized production of instances in this young literature.

### ALNS Metaheuristics:-

ALNS was introduced by Ropke and Pisinger (2006) for the pickup and delivery problem with time windows, then generalized by Pisinger and Ropke (2007). Its principle is to partially destroy a current solution using destruction operators and to repair it using repair operators, the choice of operators being adapted according to their recent performance. Sacramento et al. (2019) apply this scheme to the VRPD. Our ALNS inherits its general structure while adapting it to the ternary modeling of flights (insertion of drone flights into the tours, vehicle-drone switch).

### Problem Formulation:-

#### Data:-

Let  $C = \{1, \dots, n\}$  be the set of customers, 0 the starting depot and  $n+1$  the return depot (the origin and destination depots may be geographically identical). Let  $V = \{0\} \cup C \cup \{n+1\}$  be the set of nodes. Each customer  $j \in C$  has a time window  $[a_j, b_j]$  and a service time  $s_j$  (when served by a vehicle). A vehicle traversing arc  $(i, j)$  takes time  $t_{ij}^a = d_{ij} / v^a$  where  $d_{ij}$  is the Euclidean distance and  $v^a$  the vehicle speed. The fleet is made of  $m$  identical vehicles, each of capacity  $Q$ . On board each vehicle there are  $p$  identical drones, so that  $D = \{1 \dots mp\}$  is the set of drones; drone  $d$  is assigned to vehicle  $\kappa(d)$ . A drone travels at speed  $v^d = \sigma v^a$ ,  $\sigma > 1$  (hence  $t_{ij}^d = t_{ij}^a / \sigma$ ), carries one parcel, requires a service time  $s^d$  at the served customer and has an effective flight endurance  $E$ : the total distance of a flight  $d(i, j) + d(j, k)$  cannot exceed  $E$ . The energy cost of a flight is  $c^d(d(i, j) + d(j, k))$ .

A flight is defined by a triplet  $(i, j, k)$ : the drone is launched from node  $i$ , serves customer  $j$ , and is then recovered at node  $k$ . Nodes  $i$  and  $k$  belong to  $\{0\} \cup C$  and  $\{n+1\}$  respectively for the recovery;  $j \in C$  must be eligible for the drone. The set of candidate flights  $F$  is restricted to nearest neighbors: the launch occurs at the depot or among the  $K$  customers closest to  $j$ , the recovery among the  $K$  customers closest to  $j$  or at the depot, subject to the endurance constraint  $d(i, j) + d(j, k) \leq E$ . This choice, standard in the literature (Murray and Chu, 2015; Sacramento et al., 2019), limits the combinatorial complexity while keeping the relevant solutions; its sensitivity is discussed in Section 6.

### Costs:-

#### The total cost of a solution comprises four terms:-

6. the fixed cost  $c^f$  for each vehicle used;
7. the travel cost  $c^a$  per unit distance traveled by the vehicles;
8. the energy cost  $c^d$  per unit distance flown by the drones;
9. the penalty  $c^p$  per minute of lateness on a customer's time window.

Since the time windows are soft, the model seeks to balance the operational cost (fixed, road, energy) and the service cost (lateness). This penalty formulation is a common linear approximation of the contractual service-level clauses.

### Assumptions:-

The problem is studied under the following assumptions: unit demands (one parcel per customer), each customer served once, at most one flight per drone per tour, strict synchronization of the drone with its mothership vehicle (the vehicle waits for the drone to return before leaving the recovery node), homogeneous fleet and soft time

windows. These assumptions match the framework of the research assignment and the dominant models of the literature (Murray and Chu, 2015; Agatz et al., 2018).

### Mixed-Integer Linear Programming Model:-

#### Variables:-

$x_{ij}^v \in \{0, 1\}$  vehicle  $v$  traverses arc  $(i, j)$ ,  $i, j \in V$ ,  $i \neq j$

$y_j^v \in \{0, 1\}$  customer  $j$  is served by vehicle  $v$

$u^v \in \{0, 1\}$  vehicle  $v$  is used

$w_f^d \in \{0, 1\}$  drone  $d$  performs flight  $f = (i, j, k) \in F$

$T_i^v \geq 0$  departure time of vehicle  $v$  from node  $i$

$q_f^d \geq 0$  launch time of drone  $d$  for flight  $f$

$p_j \geq 0$  lateness of customer  $j$

#### Objective Function:-

The objective minimizes the total cost (fixed vehicle-usage, road travel, drone energy and lateness penalties):

$$\min \sum_{v=1}^m c^f u^v + \sum_{v=1}^m \sum_{i,j \in V, i \neq j} c^a d_{ij} x_{ij}^v + \sum_{d=1}^{mp} \sum_{f=(i,j,k) \in F} c^d (d(i,j) + d(j,k)) w_f^d + \sum_{j \in C} c^p p_j \quad (1)$$

Where  $m$  is the number of vehicles,  $mp$  the number of drones,  $F$  the set of candidate flights,  $d(i, j)$  the Euclidean distance between nodes  $i$  and  $j$ , and the four cost terms are defined in Section 3.2.

#### Constraints:-

**Customer coverage. Each customer is served exactly once, either by a vehicle or by a drone:-**

$$\sum_{v=1}^m y_j^v + \sum_{d=1}^{mp} \sum_{f \in F: f \text{ serves } j} w_f^d = 1, \quad \forall j \in C \quad (2)$$

**Vehicle flow. For each vehicle, flow conservation links the tour variables to  $y$  and  $u$ :-**

$$\sum_{j \in V \setminus \{i\}} x_{ij}^v = \sum_{j \in V \setminus \{i\}} x_{ji}^v = y_i^v \quad \forall v, \forall i \in C \quad \sum_{j \in CU} x_{0j}^v = u^v, \quad \sum_{i \in CU} x_{i,n+1}^v = u^v \quad \forall v \quad \sum_{j \in C} y_j^v \leq Q \quad \forall v \quad (3)$$

**Symmetry breaking. Since the vehicles are identical, we impose that the vehicles are used in order and that the number of served customers is non-increasing:-**

$$u^{v+1} \leq u^v, \quad \sum_{j \in C} y_j^{v+1} \leq \sum_{j \in C} y_j^v, \quad \forall v = 1, \dots, m-1 \quad (4)$$

**Use-flow linking (strengthening). Although redundant for any integer solution, the following linking inequalities are part of the formulation and significantly tighten the continuous relaxation:-**

$$x_{ij}^v \leq u^v \quad \forall v, i, j \in V, i \neq j, \quad y_j^v \leq u^v \quad \forall v, j \in C \quad (5)$$

**Drone-vehicle coordination. A drone performs at most one sortie; a flight is possible only if the launch node and the recovery node belong to the tour of the mothership vehicle:-**

$$\sum_{f \in F} w_f^d \leq 1 \quad \forall d \quad w_f^d \leq y_{i^{(d)}}^v, \quad w_f^d \leq y_{k^{(d)}}^v, \quad \forall d, f = (i, j, k) \in F, i, k \in C \quad (6)$$

**For drones of the same vehicle, we order their possible sorties to eliminate symmetries:-**

$$\sum_{f \in F} w_f^d \wedge \{d\} \geq \sum_{f \in F} w_f^{d'} \wedge \{d'\}, \quad \text{for each pair of consecutive drones } d, d' \text{ of the same vehicle.}$$

Temporal progression of the vehicles.

**To eliminate subtours and propagate the departure times, we impose, for every arc  $(i, j)$ :-**

$$T_j^v \geq T_i^v + t_{ij}^a + s_j - M_{ij} (1 - x_{ij}^v) \quad (7)$$

With  $M_{ij} = \max(0, b_i + P + s_i + t_{ij}^a - a_j)$  and  $P$  the maximum tolerated lateness;  $b_i$  denotes the end of the time window of customer  $i$  (or the horizon  $H$  for the depot),  $s_i$  its service time, and the departure from the depot is fixed to 0:  $T_{v_0}^v = 0$ . This tightened big- $M$  choice significantly improves the continuous relaxation.

Soft time windows.

**For a customer  $j$  served by vehicle  $v$ , the effective arrival is  $T_{v_j}^v - s_j$ :-**

$$T_j^v - s_j \geq a_j - M' (1 - y_j^v) \quad T_j^v - s_j \leq b_j + p_j + M' (1 - y_j^v) \quad (8)$$

With  $M' = H + \max_j s_j + P$ .

**Vehicle-drone synchronization. The drone is launched upon the arrival of the mothership vehicle at the launch node, and the vehicle waits for its return before leaving. For flight  $f = (i, j, k)$  of drone  $d$ :-**

$$q_f^d \geq T_i^{k(d)} - s_i - M_f(1 - w_f^d) \quad \text{if } i \in C \quad T_k^{k(d)} \geq q_f^d + t_{ij}^d + s^d + t_{jk}^d - M_f(1 - w_f^d) \quad q_f^d + t_{ij}^d \geq a_j - M_f(1 - w_f^d) \quad q_f^d + t_{ij}^d \leq b_j + p_j + M_f(1 - w_f^d) \quad (9)$$

With  $M_f = \max(0, b_i + P + s_i) + t_{ij}^d + s^d + t_{jk}^d$ . The first constraint expresses that the launch occurs upon the arrival of the vehicle at  $i$ ; the second that the recovery at  $k$  can only take place after the end of the flight; the last two impose the (soft) time windows of the customer served by the drone.

**Subtour elimination (DFJ).** Although the time progression (7) eliminates subtours in the integer model, we strengthen the continuous relaxation with DFJ inequalities separated dynamically:-

$$\sum_{v=1}^m \sum_{i,j \in S, i \neq j} x_{ij}^v \leq |S| - 1, \quad \forall S \subset C, |S| \geq 2 \quad (10)$$

The separation is performed on the connected components of the support graph of the fractional solution (without the depot), which provides valid and aggressive cuts at the root.

#### Lower Bound:-

The lower bound  $\underline{z}$  is obtained by solving the continuous relaxation of the model, strengthened iteratively by the cuts (10) (until convergence) together with the linking inequalities (5), which are activated at a negligible computational cost. This bound is used, on the one hand, to quantify the quality of the ALNS solutions on the instances solved to optimality, and on the other hand, to provide a quality indicator on large instances when the exact model has not converged within the time limit. Table 2 reports the root bound value  $\underline{z}_0$ , illustrating the effect of the strengthening at the root.

**Proposition 1. Inequality (10) is valid for any feasible integer solution of the VRPTW-D:-**

Proof. Let  $S \subset C$ ,  $|S| \geq 2$ , and consider a feasible solution. The total number of arcs internal to  $S$  used by all vehicles is  $\sum_v x^v(S, S)$ . Each vehicle crossing  $S$  visits a sub-sequence of customers of  $S$ ; for such a sub-sequence of cardinality  $r \geq 1$ , the number of internal arcs is  $r - 1$  if the vehicle enters and leaves  $S$  (the depot departure/return case included). Summing over all vehicles, the total number of internal arcs is  $|S| - \eta$ , where  $\eta \geq 1$  is the total number of entries into  $S$ . Consequently  $\sum_v x^v(S, S) \leq |S| - 1$ .

**Proposition 2. Inequalities (5) are valid for any feasible integer solution of the VRPTW-D:-**

Proof. If an arc  $(i, j)$  is traversed by vehicle  $v$ , then  $u^v = 1$  by definition of the use variables, hence  $x^v_{ij} \leq 1 = u^v$ ; if  $u^v = 0$ , vehicle  $v$  serves no customer ( $y^v_j = 0$  for all  $j$ ) and traverses no arc. The claim follows.

#### ALNS Heuristic:-

##### General Principle:-

The ALNS builds an initial solution by a greedy insertion sorted by time-window end, then iteratively improves a current solution by applying a destruction operator (removal of  $q$  customers) and then a repair operator (reinsertion of the removed customers). The (destruction, repair) pair is drawn according to adaptive weights updated by segments. A neighbor solution is accepted if it is better than the current solution, or according to a simulated-annealing criterion with a geometrically decreasing temperature. A local search is applied periodically and at the end of the algorithm.

##### Solution Representation:-

A solution is a collection of tours. A tour is defined by an ordered sequence of stops (starting depot 0, customers, return depot  $n+1$ ) and by a list of flights anchored in the sequence: a flight  $(i, j, k)$  is anchored at the position of the launch node  $i$  and at that of the recovery node  $k$ . The evaluation of a tour propagates time: the vehicle leaves the depot at 0, the drone is launched upon the arrival of the vehicle at the launch node, the vehicle waits for the return of the drone before leaving the recovery node, and the cost accumulates the road distance, the flight energy, the lateness penalties and, per tour, the fixed vehicle cost.

##### Destruction Operators:-

- **Random destruction:**  $q$  customers are removed uniformly at random.
- **Worst destruction:** the customers are ranked by removal gain (decrease of the total cost if removed) and the  $q$  best gains are removed, with random noise.
- **Shaw destruction:** one customer is removed, then the next ones are chosen among the closest according to a combination of distance, time-window gap and service type (vehicle/drone).

- **Flight destruction:** a drone flight is selected and its served customer is removed, together with the neighboring customers.

#### Repair Operators:-

- **Greedy repair:** the customers are reinserted one by one at their best position (best arc of a tour, or best admissible drone flight).
- **Regret-2 repair:** for each customer, the cost of its best and second-best insertions is computed; the customer maximizing the difference (regret) is inserted first, which favors the most constrained customers.

The computation of the insertion options integrates the possible creation of a new tour (with its fixed cost) and verifies that each proposed flight belongs to the candidate set F: the ALNS thus explores exactly the same solution space as the exact model.

#### Local Search:-

Three local-search moves are applied: the 2-opt exchange on the sequences of a tour, the relocation of a customer between tours, and the vehicle-drone switch (a customer served by a vehicle becomes a candidate for a drone flight and conversely), subject to admissibility.

#### Calibration:-

The ALNS is run with 4 destruction operators, 2 repair operators, an initial temperature  $T_0 = 0.25 z_{init}$ , a cooling factor  $\rho = 0.995$  updated every 100 iterations, and a number of removed customers  $q \in [\max(2, 0.2n), \max(3, 0.5n)]$ . The parameters (adaptive weights, acceptance threshold, local-search frequency) are calibrated on the small instances whose optimal solutions are known (Section 6).

#### Experimental Study:-

##### Instances:-

The instances are generated in the style of Solomon (1987) according to three classes:-

- **C1:** customers clustered in five clusters (polarized geographical structure, wide time windows covering the whole horizon);
- **R1:** uniformly distributed customers, narrow time windows;
- **RC1:** a mix of clustered and dispersed zones.

Each instance is defined by the number  $n$  of customers, a class, a generation seed and a time-window mode (tight, medium and wide). The positions are generated in a  $100 \times 100$  km square centered at the depot (40, 40). The vehicle speed is  $v^a = 1$  km per time unit (60 km/h), the drone speed ratio  $\sigma = 2$ , the effective endurance  $E = 15$  km (a manufacturer endurance of 20 km multiplied by a degradation factor of 0.75), the service times  $s_j = 10$  and  $s^d = 2$  minutes, and the costs  $c^f = 50$ ,  $c^a = 0.75$ ,  $c^d = 0.15$ ,  $c^p = 1$  (\$/min). The capacity is set to  $Q = \lceil n/m \rceil$  with  $m = \max(2, \lceil n/10 \rceil)$  vehicles and 2 drones per vehicle. About 50% of the customers are eligible for the drone (coverage rate), this rate being drawn at random.

#### Resolution Settings:-

The exact models are solved with CPLEX 22.1 (full Studio) via the Python interface. The parameters are the following: time limits of 120 s ( $n = 10$ ), 300 s ( $n = 15$ ), 600 s ( $n = 25$ ) and 900 s ( $n = 50$ ), gap tolerance of  $10^{-4}$  for  $n \leq 15$  and  $10^{-3}$  beyond, DFJ cut separation at the root, 4 threads. The strengthened model (use-flow linking, Section 4) is used throughout, and column  $\underline{z}_0$  of Table 2 reports the root lower bound (continuous relaxation plus DFJ separation and linking inequalities). On two  $n = 25$  instances the search exceeded the nominal clock: C25\_s3, restarted after CPLEX returned no incumbent within the first 600 s, took 1124 s in total, and R25\_s2 a single run of 8621 s stalled under memory pressure; both times are reported as is. The four  $n = 10$  instances not closed in 120 s have been re-run with a 600 s limit and an ALNS warm start (feasible incumbent). The ALNS is run with time limits of 30 s ( $n = 10$ ), 45 s ( $n = 15$ ), 60 s ( $n = 25$ ), 90 s ( $n = 50$ ) and 120 s ( $n = 100$ ), for three random seeds.

#### Results: Exact Model and ALNS:-

Table 2 reports the exact results. The nine instances of size  $n = 10$  are all solved to optimality: the clustered instances C10\_s1-s3 within 17-47 s and R10\_s3, RC10\_s3 within 65-68 s. The four dispersed instances R10\_s1, R10\_s2, RC10\_s1, RC10\_s2 were not closed at the initial 120 s limit (gap of 17-19% at the stop); re-run with a 600 s limit and an ALNS warm start, they are all proven optimal within 46-117 s, the search tightening the lower bound up to optimality. Beyond  $n = 10$ , no optimality is proven within the time limits: the relative gaps are 36-64% at  $n = 15$  (300 s limit), 80-94% at  $n = 25$  (600 s) and 85-99% at  $n = 50$  (900 s). The strengthening inequalities of Section 4

lift the root bound (column  $\underline{z}_0$ ) well above the continuous relaxation of the bare model: at  $n = 15$ , the root alone already represents between 39% and 80% of the bound reached after 300 s of branch-and-bound, and the final bounds close 36-64% of the optimum value. On C15\_s1, for instance, the root bound is 109.7 and the bound after 300 s is 193.5, i.e. 64% of  $z^* = 304.0$ . The lower bound  $\underline{z}$  delivered by the compact formulation nevertheless remains weak from  $n \geq 25$  on, a classical limitation of VRPTW-D models. The incumbents of the  $n = 50$  instances of the R and RC classes are obtained by an ALNS warm start: on these instances, CPLEX heuristics found no integer solution within the time limit, so a feasible incumbent was provided by the ALNS before resuming the search. The column Base gives the cost of the optimal VRPTW solution without drones, which isolates the economic contribution of the drones.

**Table 2: Exact results (CPLEX 22.1) within the stated time limit.  $z^*$ : value of the best solution; Opt.: ✓ when optimality is proven (otherwise -);  $\underline{z}$ : best lower bound found by the solver;  $\underline{z}_0$ : root bound (continuous relaxation strengthened by the DFJ inequalities and the use/arc linking); Gap: relative gap (%); Time: (s); Nodes: branch-and-bound nodes; Dr.: number of drone flights; Base: total cost without drones (VRPTW).**

| Inst.   | Cl. | n  | $z^*$   | Opt. | $\underline{z}$ | $\underline{z}_0$ | Gap   | Time   | Nodes  | Dr. | Base                                         |
|---------|-----|----|---------|------|-----------------|-------------------|-------|--------|--------|-----|----------------------------------------------|
| C10_s1  | C   | 10 | 408.63  | ✓    | 408.59          | 123.52            | 0.01  | 26.5   | 54733  | 3   | 527.15                                       |
| C10_s2  | C   | 10 | 358.21  | ✓    | 358.18          | 136.62            | 0.01  | 16.9   | 23385  | 3   | 422.35                                       |
| C10_s3  | C   | 10 | 434.16  | ✓    | 434.12          | 160.76            | 0.01  | 46.7   | 110850 | 3   | 533.29                                       |
| C15_s1  | C   | 15 | 304.01  | -    | 193.48          | 109.74            | 36.36 | 300.3  | 113808 | 4   | 408.8                                        |
| C15_s2  | C   | 15 | 548.77  | -    | 232.4           | 89.93             | 57.65 | 300.2  | 184899 | 4   | 659.27                                       |
| C15_s3  | C   | 15 | 336.77  | -    | 207.11          | 108.44            | 38.5  | 300.2  | 193041 | 4   | 481.23                                       |
| C25_s1  | C   | 25 | 1587.94 | -    | 158.53          | 109.67            | 90.02 | 600.8  | 248681 | 3   | 1835.78                                      |
| C25_s2  | C   | 25 | 791.16  | -    | 123.85          | 96.43             | 84.35 | 600.4  | 329283 | 4   | 1153.16                                      |
| C25_s3  | C   | 25 | 1848.74 | -    | 119.81          | 101.86            | 93.52 | 1123.9 | 186827 | 4   | 1812.78                                      |
| C50_s1  | C   | 50 | 3930.35 | -    | 27.63           | 76.43             | 99.3  | 904.7  | 6009   | 7   | 2883.49                                      |
| C50_s2  | C   | 50 | 2160.9  | -    | 23.7            | 73.27             | 98.9  | 908.5  | 5962   | 10  | 3285.26                                      |
| C50_s3  | C   | 50 | 3170.45 | -    | 25.47           | 73.9              | 99.2  | 908.0  | 5549   | 5   | 2896.79                                      |
| R10_s1  | R   | 10 | 691.13  | ✓    | 691.13          | 309.11            | 0.0   | 46.0   | 96612  | 0   | 691.13                                       |
| R10_s2  | R   | 10 | 578.44  | ✓    | 578.4           | 277.08            | 0.01  | 82.0   | 175673 | 0   | 578.44                                       |
| R10_s3  | R   | 10 | 577.05  | ✓    | 577.03          | 287.87            | 0.0   | 67.9   | 96348  | 0   | 577.05                                       |
| R15_s1  | R   | 15 | 1168.28 | -    | 423.11          | 339.88            | 63.78 | 300.1  | 408263 | 1   | 1227.7                                       |
| R15_s2  | R   | 15 | 919.69  | -    | 415.17          | 274.12            | 54.86 | 300.1  | 337065 | 2   | 949.24                                       |
| R15_s3  | R   | 15 | 1044.97 | -    | 383.29          | 297.91            | 63.32 | 300.1  | 312047 | 1   | 1115.87                                      |
| R25_s1  | R   | 25 | 2486.59 | -    | 453.1           | 379.09            | 81.78 | 600.2  | 375284 | 3   | 2909.31                                      |
| R25_s2  | R   | 25 | 2276.79 | -    | 442.79          | 361.29            | 80.55 | 8621.2 | 301888 | 1   | 2607.37                                      |
| R25_s3  | R   | 25 | 2055.3  | -    | 419.49          | 335.34            | 79.59 | 601.6  | 268561 | 4   | 2511.37                                      |
| R50_s1  | R   | 50 | 2948.89 | -    | 431.68          | 448.62            | 85.36 | 1053.7 | 9773   | 8   | 3768.94                                      |
| R50_s2  | R   | 50 | 2808.07 | -    | 379.01          | 398.4             | 86.5  | 1001.1 | 11255  | 9   | 3824.9                                       |
| R50_s3  | R   | 50 | 2847.48 | -    | 321.6           | 367.14            | 88.71 | 1012.5 | 4652   | 7   | 5639.32                                      |
| RC10_s1 | RC  | 10 | 706.18  | ✓    | 706.17          | 301.1             | 0.0   | 114.2  | 229997 | 0   | 706.18                                       |
| RC10_s2 | RC  | 10 | 590.75  | ✓    | 590.72          | 295.32            | 0.01  | 117.0  | 186367 | 0   | 590.75                                       |
| RC10_s3 | RC  | 10 | 460.69  | ✓    | 460.65          | 232.3             | 0.01  | 65.1   | 79025  | 1   | 484.95                                       |
| RC15_s1 | RC  | 15 | 1017.15 | -    | 401.26          | 281.92            | 60.55 | 300.1  | 269634 | 3   | 1130.86                                      |
| RC15_s2 | RC  | 15 | 848.37  | -    | 382.75          | 281.83            | 54.88 | 300.1  | 279024 | 2   | 975.83                                       |
| RC15_s3 | RC  | 15 | 1080.7  | -    | 434.92          | 256.72            | 59.76 | 300.1  | 327033 | 3   | 1224.61                                      |
| RC25_s1 | RC  | 25 | 1997.41 | -    | 359.96          | 285.59            | 81.98 | 600.7  | 325508 | 4   | 2702.38                                      |
| RC25_s2 | RC  | 25 | 2057.13 | -    | 371.6           | 292.4             | 81.94 | 600.4  | 194031 | 4   | 2300.81                                      |
| RC25_s3 | RC  | 25 | 2044.71 | -    | 320.77          | 256.98            | 84.31 | 601.0  | 196720 | 2   | 2293.35                                      |
| RC50_s1 | RC  | 50 | 2625.86 | -    | 334.26          | 365.61            | 87.27 | 1033.1 | 5906   | 8   | 3354.45                                      |
| RC50_s2 | RC  | 50 | 2515.82 | -    | 292.87          | 335.03            | 88.36 | 1022.3 | 3549   | 9   | 3365.26                                      |
| RC50_s3 | RC  | 50 | 2494.64 | -    | 311.55          | 326.47            | 87.51 | 1096.2 | 6936   | 9   | 3219.86                                      |
|         |     |    |         |      |                 |                   |       |        |        |     | Optimality:<br>9/36<br>-<br>average<br>cost: |

|  |  |  |  |  |  |  |  |  |  |         |
|--|--|--|--|--|--|--|--|--|--|---------|
|  |  |  |  |  |  |  |  |  |  | 1520.06 |
|--|--|--|--|--|--|--|--|--|--|---------|

Table 3 summarizes the economic contribution of the drones. All the classes select drone flights: on average 4.5 flights per instance on C, 3.0 on R and 3.8 on RC. The average relative gain with respect to the drone-free solution is 27.6% on C (between 15 and 49%), 15.1% on R and 15.0% on RC, and it grows with the instance size (6.8% at  $n = 10$ , 14.0% at  $n = 15$ , 22.4% at  $n = 25$ , 33.7% at  $n = 50$ ). On C10\_s1, the total cost drops from 527.15 (without drones) to 408.63, i.e. -22.5%, with three drone flights. The drone thus yields a significant gain on every class, all the more pronounced as the road fleet is stretched.

**Table 3: Economic contribution of the drones: total cost with drones ( $z_{dr}$ ), without drones ( $z_{sd}$ , VRPTW) and relative gain.**

| Inst.   | Cl. | n  | $z_{dr}$ | $z_{sd}$ | Gain (%)   | Flights |
|---------|-----|----|----------|----------|------------|---------|
| C10_s1  | C   | 10 | 408.63   | 527.15   | 22.48      | 3       |
| C10_s2  | C   | 10 | 358.21   | 422.35   | 15.19      | 3       |
| C10_s3  | C   | 10 | 434.16   | 533.29   | 18.59      | 3       |
| C15_s1  | C   | 15 | 304.01   | 408.8    | 25.63      | 4       |
| C15_s2  | C   | 15 | 548.77   | 659.27   | 16.76      | 4       |
| C15_s3  | C   | 15 | 336.77   | 481.23   | 30.02      | 4       |
| C25_s1  | C   | 25 | 1296.96  | 1835.78  | 29.35      | 3       |
| C25_s2  | C   | 25 | 791.16   | 1153.16  | 31.39      | 4       |
| C25_s3  | C   | 25 | 1484.12  | 1812.78  | 18.13      | 4       |
| C50_s1  | C   | 50 | 1872.08  | 2883.49  | 35.08      | 7       |
| C50_s2  | C   | 50 | 1682.8   | 3285.26  | 48.78      | 10      |
| C50_s3  | C   | 50 | 1742.79  | 2896.79  | 39.84      | 5       |
| R10_s1  | R   | 10 | 691.13   | 691.13   | 0.0        | 0       |
| R10_s2  | R   | 10 | 578.44   | 578.44   | 0.0        | 0       |
| R10_s3  | R   | 10 | 577.05   | 577.05   | 0.0        | 0       |
| R15_s1  | R   | 15 | 1168.28  | 1227.7   | 4.84       | 1       |
| R15_s2  | R   | 15 | 904.45   | 949.24   | 4.72       | 2       |
| R15_s3  | R   | 15 | 1024.24  | 1115.87  | 8.21       | 1       |
| R25_s1  | R   | 25 | 2331.52  | 2909.31  | 19.86      | 3       |
| R25_s2  | R   | 25 | 2105.02  | 2607.37  | 19.27      | 1       |
| R25_s3  | R   | 25 | 2055.3   | 2511.37  | 18.16      | 4       |
| R50_s1  | R   | 50 | 2948.89  | 3768.94  | 21.76      | 8       |
| R50_s2  | R   | 50 | 2623.76  | 3824.9   | 31.4       | 9       |
| R50_s3  | R   | 50 | 2642.15  | 5639.32  | 53.15      | 7       |
| RC10_s1 | RC  | 10 | 706.18   | 706.18   | -0.0       | 0       |
| RC10_s2 | RC  | 10 | 590.75   | 590.75   | -0.0       | 0       |
| RC10_s3 | RC  | 10 | 460.69   | 484.95   | 5.0        | 1       |
| RC15_s1 | RC  | 15 | 1004.84  | 1130.86  | 11.14      | 3       |
| RC15_s2 | RC  | 15 | 848.37   | 975.83   | 13.06      | 2       |
| RC15_s3 | RC  | 15 | 1080.7   | 1224.61  | 11.75      | 3       |
| RC25_s1 | RC  | 25 | 1997.41  | 2702.38  | 26.09      | 4       |
| RC25_s2 | RC  | 25 | 2057.13  | 2300.81  | 10.59      | 4       |
| RC25_s3 | RC  | 25 | 1631.36  | 2293.35  | 28.87      | 2       |
| RC50_s1 | RC  | 50 | 2508.04  | 3354.45  | 25.23      | 8       |
| RC50_s2 | RC  | 50 | 2515.82  | 3365.26  | 25.24      | 9       |
| RC50_s3 | RC  | 50 | 2494.64  | 3219.86  | 22.52      | 9       |
|         |     |    |          |          | Average C  | 27.60   |
|         |     |    |          |          | Average R  | 15.11   |
|         |     |    |          |          | Average RC | 14.96   |

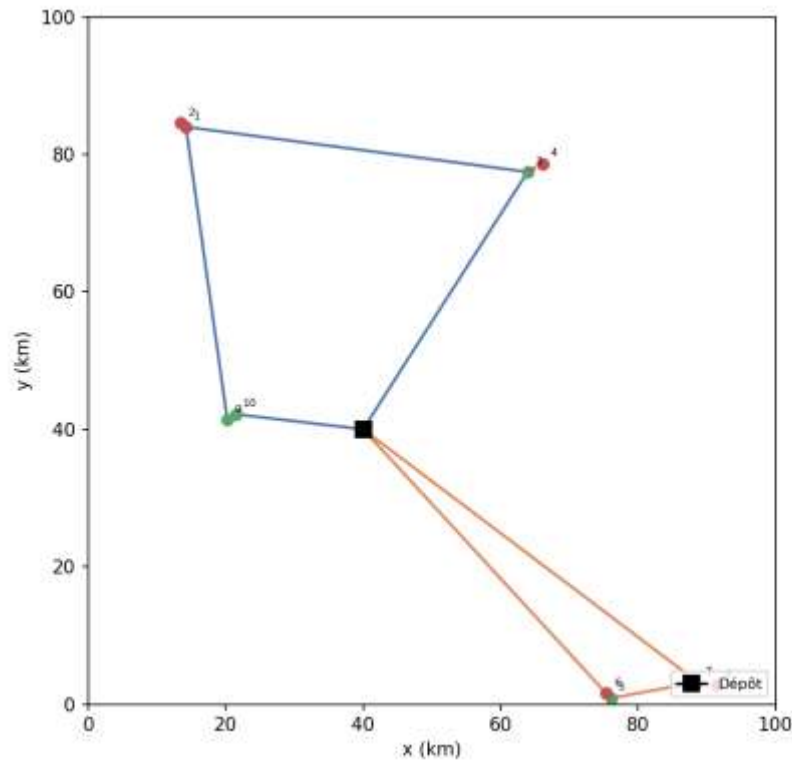


**Figure 1: Optimal solution of C10\_s1: two vehicle tours (solid lines, one color per tour) and three drone flights (dashed red segments). Customer labels appear next to the points; the square is the depot.**

Figure 1 displays the optimal solution of C10\_s1. Each drone flight  $(i, j, k)$  relieves the vehicle tour of the customer  $j$  while keeping the launch and recovery nodes on the mothership tour.

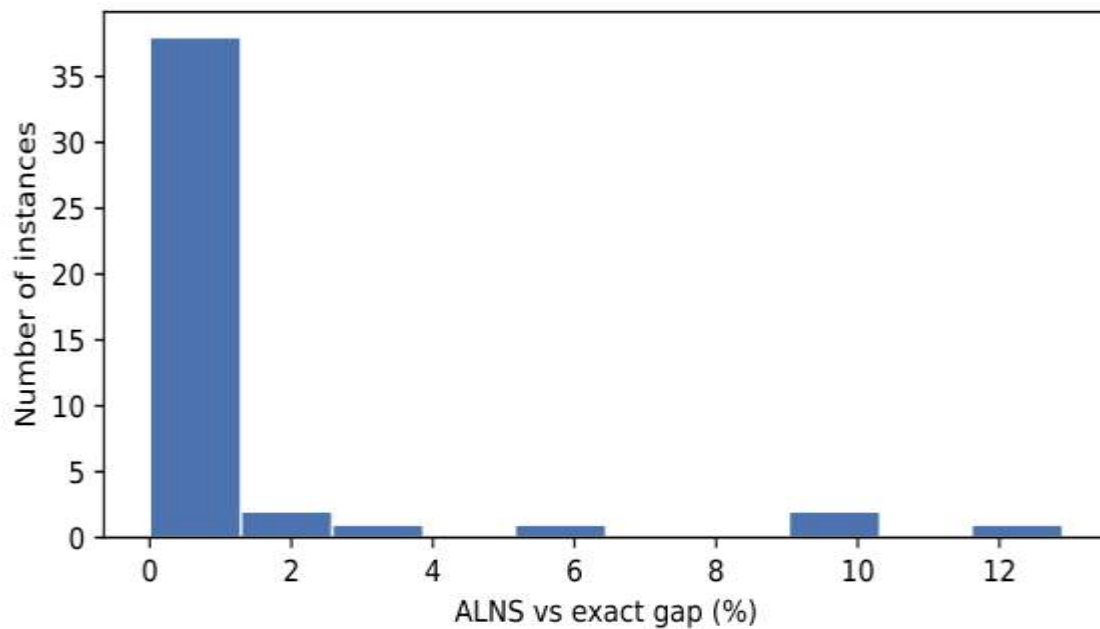


**Figure 2: Distribution of the relative gaps of the best ALNS run against the best known  $z^*$  over the 45 instances.**

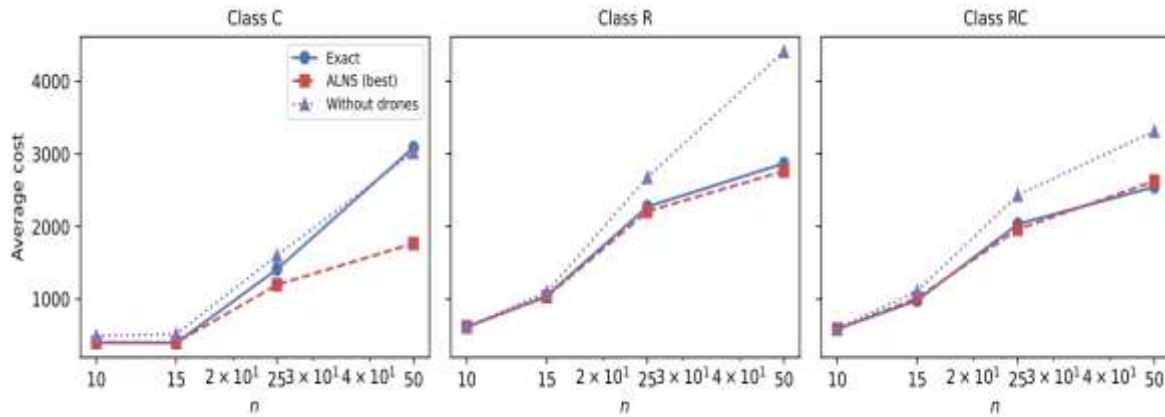


**Figure 3: Average cost as a function of the number of customers  $n$ , by class: exact solutions, best ALNS and drone-free baseline.**

Tables 4 and 5 report the ALNS results, assessed against the best known  $z^*$ : exact optimum when proven, otherwise the minimum of the exact incumbent and the best ALNS value over the three seeds. Over the 9 instances of size  $n = 10$ , now all solved to optimality, the best seed reproduces the exact optimum on 7 instances and stays within 1% on the other two (C10\_s2 +0.96%, C10\_s3 +0.94%), i.e. an average gap of 0.21%. Over the whole set of 45 instances, the average gap of the best seed to the best known is 1.14%, below 2% on 38 instances (84%). The seven gaps above 2%, up to 12.9% on RC50\_s3 (and 9.9% on RC15\_s3), correspond to instances where the exact incumbent is markedly better than the best ALNS solution found within the time budget. The standard deviation across the three seeds remains small (Table 5), which indicates a robust and reproducible heuristic.

Figure 2 shows the distribution of the relative gaps of the best ALNS run against the best known, and Figure 3 compares the average cost of the exact solutions, of the ALNS and of the drone-free baseline as a function of the instance size. Beyond  $n = 50$  the exact campaign is not carried out (900 s limit reached with weak bounds) and the  $n = 100$  instances are handled by the ALNS only: for these sizes, the best known is the ALNS solution. On the larger instances ( $n \geq 15$ ), the exact model does not prove optimality within the time limits but always delivers a feasible solution together with the lower bound  $\underline{z}$  of Table 2; the ALNS, whose gap to the best known remains below 2% on 84% of the instances, is the method used for the largest sizes ( $n = 50, 100$ ).

**Table 4: ALNS results (best of 3 seeds).  $z^*$ : best known (exact optimum when proven, otherwise the minimum of the exact incumbent and the best ALNS value; \*: reference obtained by the ALNS). Gap: relative deviation (%) from the best known.**

| Inst.   | Cl. | n   | ALNS    | Time  | It.  | Dr. | $z^*$    | Gap (%) |
|---------|-----|-----|---------|-------|------|-----|----------|---------|
| C100_s1 | C   | 100 | 3614.4  | 122.4 | 47   | 16  | 3614.40* | 0.0     |
| C100_s2 | C   | 100 | 4683.21 | 121.0 | 410  | 14  | 4683.21* | 0.0     |
| C100_s3 | C   | 100 | 3528.81 | 123.1 | 65   | 16  | 3528.81* | 0.0     |
| C10_s1  | C   | 10  | 408.63  | 30.3  | 4903 | 3   | 408.63   | 0.0     |
| C10_s2  | C   | 10  | 361.65  | 30.1  | 8540 | 3   | 358.21   | 0.96    |
| C10_s3  | C   | 10  | 438.22  | 30.2  | 7467 | 3   | 434.16   | 0.94    |
| C15_s1  | C   | 15  | 314.71  | 45.5  | 1651 | 3   | 304.01   | 3.52    |
| C15_s2  | C   | 15  | 550.24  | 45.2  | 1815 | 4   | 548.77   | 0.27    |
| C15_s3  | C   | 15  | 339.07  | 45.3  | 4325 | 2   | 336.77   | 0.68    |
| C25_s1  | C   | 25  | 1296.96 | 60.5  | 1627 | 4   | 1296.96* | 0.0     |
| C25_s2  | C   | 25  | 810.1   | 60.6  | 1910 | 4   | 791.16   | 2.39    |
| C25_s3  | C   | 25  | 1484.12 | 60.6  | 1770 | 4   | 1484.12* | 0.0     |
| C50_s1  | C   | 50  | 1872.08 | 93.8  | 140  | 8   | 1872.08* | 0.0     |
| C50_s2  | C   | 50  | 1682.8  | 93.8  | 135  | 7   | 1682.80* | 0.0     |
| C50_s3  | C   | 50  | 1742.79 | 93.3  | 140  | 8   | 1742.79* | 0.0     |
| R100_s1 | R   | 100 | 4948.91 | 122.4 | 30   | 15  | 4948.91* | 0.0     |
| R100_s2 | R   | 100 | 5417.58 | 121.8 | 230  | 16  | 5417.58* | 0.0     |

|          |    |     |         |       |       |    |          |       |
|----------|----|-----|---------|-------|-------|----|----------|-------|
| R100_s3  | R  | 100 | 5253.01 | 121.6 | 55    | 14 | 5253.01* | 0.0   |
| R10_s1   | R  | 10  | 691.13  | 30.1  | 24295 | 0  | 691.13   | 0.0   |
| R10_s2   | R  | 10  | 578.44  | 30.1  | 25425 | 0  | 578.44   | 0.0   |
| R10_s3   | R  | 10  | 577.05  | 30.1  | 25590 | 0  | 577.05   | 0.0   |
| R15_s1   | R  | 15  | 1168.28 | 45.3  | 11545 | 1  | 1168.28  | 0.0   |
| R15_s2   | R  | 15  | 904.45  | 45.4  | 2694  | 2  | 904.45*  | 0.0   |
| R15_s3   | R  | 15  | 1024.24 | 45.7  | 7085  | 1  | 1024.24* | 0.0   |
| R25_s1   | R  | 25  | 2331.52 | 60.5  | 4335  | 2  | 2331.52* | 0.0   |
| R25_s2   | R  | 25  | 2105.02 | 60.8  | 2800  | 2  | 2105.02* | 0.0   |
| R25_s3   | R  | 25  | 2186.59 | 61.0  | 1021  | 4  | 2055.3   | 6.39  |
| R50_s1   | R  | 50  | 3024.44 | 93.8  | 208   | 8  | 2948.89  | 2.56  |
| R50_s2   | R  | 50  | 2623.76 | 91.9  | 77    | 10 | 2623.76* | 0.0   |
| R50_s3   | R  | 50  | 2642.15 | 92.2  | 950   | 8  | 2642.15* | 0.0   |
| RC100_s1 | RC | 100 | 4923.56 | 120.7 | 745   | 14 | 4923.56* | 0.0   |
| RC100_s2 | RC | 100 | 5249.13 | 121.5 | 151   | 14 | 5249.13* | 0.0   |
| RC100_s3 | RC | 100 | 4555.91 | 122.2 | 114   | 15 | 4555.91* | 0.0   |
| RC10_s1  | RC | 10  | 706.18  | 30.1  | 24888 | 0  | 706.18   | -0.0  |
| RC10_s2  | RC | 10  | 590.75  | 30.1  | 25145 | 0  | 590.75   | 0.0   |
| RC10_s3  | RC | 10  | 460.69  | 30.7  | 7810  | 1  | 460.69   | -0.0  |
| RC15_s1  | RC | 15  | 1004.84 | 45.4  | 4610  | 2  | 1004.84* | 0.0   |
| RC15_s2  | RC | 15  | 848.37  | 45.4  | 2690  | 2  | 848.37   | 0.0   |
| RC15_s3  | RC | 15  | 1187.49 | 45.5  | 10840 | 0  | 1080.7   | 9.88  |
| RC25_s1  | RC | 25  | 2000.93 | 60.7  | 1355  | 3  | 1997.41  | 0.18  |
| RC25_s2  | RC | 25  | 2252.12 | 60.5  | 4429  | 0  | 2057.13  | 9.48  |
| RC25_s3  | RC | 25  | 1631.36 | 60.7  | 1195  | 4  | 1631.36* | 0.0   |
| RC50_s1  | RC | 50  | 2508.04 | 92.2  | 470   | 7  | 2508.04* | 0.0   |
| RC50_s2  | RC | 50  | 2547.29 | 93.2  | 185   | 8  | 2515.82  | 1.25  |
| RC50_s3  | RC | 50  | 2816.32 | 93.6  | 640   | 2  | 2494.64  | 12.89 |

**Table 5: ALNS summary over 3 seeds: best value, average, standard deviation, and deviation of the best value from the best known.**

| Inst.   | Cl. | n   | Best    | Avg     | Std dev | Gap (%) |
|---------|-----|-----|---------|---------|---------|---------|
| C100_s1 | C   | 100 | 3614.4  | 4188.73 | 468.43  | 0.0     |
| C100_s2 | C   | 100 | 4683.21 | 4893.36 | 152.84  | 0.0     |
| C100_s3 | C   | 100 | 3528.81 | 3610.06 | 80.89   | 0.0     |
| C10_s1  | C   | 10  | 408.63  | 408.63  | 0.0     | 0.0     |
| C10_s2  | C   | 10  | 361.65  | 361.78  | 0.18    | 0.96    |
| C10_s3  | C   | 10  | 438.22  | 445.22  | 7.04    | 0.94    |
| C15_s1  | C   | 15  | 314.71  | 314.89  | 0.13    | 3.52    |
| C15_s2  | C   | 15  | 550.24  | 558.18  | 5.98    | 0.27    |
| C15_s3  | C   | 15  | 339.07  | 347.44  | 11.02   | 0.68    |
| C25_s1  | C   | 25  | 1296.96 | 1307.02 | 8.91    | 0.0     |
| C25_s2  | C   | 25  | 810.1   | 890.25  | 56.86   | 2.39    |
| C25_s3  | C   | 25  | 1484.12 | 1503.27 | 19.9    | 0.0     |
| C50_s1  | C   | 50  | 1872.08 | 1937.44 | 48.5    | 0.0     |
| C50_s2  | C   | 50  | 1682.8  | 1797.75 | 109.57  | 0.0     |
| C50_s3  | C   | 50  | 1742.79 | 1756.33 | 10.59   | 0.0     |
| R100_s1 | R   | 100 | 4948.91 | 5209.05 | 185.58  | 0.0     |
| R100_s2 | R   | 100 | 5417.58 | 5629.37 | 149.99  | 0.0     |
| R100_s3 | R   | 100 | 5253.01 | 5356.32 | 87.1    | 0.0     |
| R10_s1  | R   | 10  | 691.13  | 691.13  | 0.0     | 0.0     |
| R10_s2  | R   | 10  | 578.44  | 584.07  | 4.06    | 0.0     |
| R10_s3  | R   | 10  | 577.05  | 577.05  | 0.0     | 0.0     |
| R15_s1  | R   | 15  | 1168.28 | 1171.17 | 4.09    | 0.0     |
| R15_s2  | R   | 15  | 904.45  | 909.19  | 4.59    | 0.0     |

|          |    |     |         |         |        |       |
|----------|----|-----|---------|---------|--------|-------|
| R15 s3   | R  | 15  | 1024.24 | 1032.66 | 7.34   | 0.0   |
| R25 s1   | R  | 25  | 2331.52 | 2423.94 | 65.47  | 0.0   |
| R25 s2   | R  | 25  | 2105.02 | 2173.47 | 48.52  | 0.0   |
| R25 s3   | R  | 25  | 2186.59 | 2251.18 | 59.32  | 6.39  |
| R50 s1   | R  | 50  | 3024.44 | 3130.12 | 95.37  | 2.56  |
| R50 s2   | R  | 50  | 2623.76 | 2722.36 | 71.71  | 0.0   |
| R50 s3   | R  | 50  | 2642.15 | 2777.9  | 96.0   | 0.0   |
| RC100 s1 | RC | 100 | 4923.56 | 4988.67 | 91.75  | 0.0   |
| RC100 s2 | RC | 100 | 5249.13 | 5298.17 | 64.23  | 0.0   |
| RC100 s3 | RC | 100 | 4555.91 | 4776.35 | 193.98 | 0.0   |
| RC10 s1  | RC | 10  | 706.18  | 707.77  | 2.24   | -0.0  |
| RC10 s2  | RC | 10  | 590.75  | 592.62  | 2.65   | 0.0   |
| RC10 s3  | RC | 10  | 460.69  | 462.37  | 2.39   | -0.0  |
| RC15 s1  | RC | 15  | 1004.84 | 1017.25 | 17.56  | 0.0   |
| RC15 s2  | RC | 15  | 848.37  | 855.01  | 9.39   | 0.0   |
| RC15 s3  | RC | 15  | 1187.49 | 1215.95 | 20.13  | 9.88  |
| RC25 s1  | RC | 25  | 2000.93 | 2014.84 | 10.16  | 0.18  |
| RC25 s2  | RC | 25  | 2252.12 | 2335.24 | 73.34  | 9.48  |
| RC25 s3  | RC | 25  | 1631.36 | 1660.48 | 27.05  | 0.0   |
| RC50 s1  | RC | 50  | 2508.04 | 2650.75 | 105.34 | 0.0   |
| RC50 s2  | RC | 50  | 2547.29 | 2628.43 | 70.0   | 1.25  |
| RC50 s3  | RC | 50  | 2816.32 | 2986.22 | 121.56 | 12.89 |

Reading the ALNS gaps. The reference  $z^*$  of Table 4 is the exact incumbent, whose optimality is not proven from  $n \geq 15$  on, and the solver gaps remain large (80-94% at  $n = 25$ , 85-9% at  $n = 50$ ). An ALNS solution lying within the proof interval therefore displays a zero gap without being provably optimal, while the deviations reported against the incumbents overestimate the distance to the true optimum on the larger sizes. Where the exact reference is the tightest, i.e. on the  $n \leq 10$  instances (all proven optimal) and on the  $n = 15$  ones, the average gap of the best seed is 0.90% (and only 0.21% at  $n = 10$ ): the parameters of the algorithm were calibrated on precisely those sizes.

#### Sensitivity Analysis:-

The sensitivity is evaluated around the reference instance  $n = 15$ , class C, seed 1, medium windows, endurance  $E = 15$  km and speed ratio  $\sigma = 2$ , each factor (endurance, speed ratio, time-window width) varies one at a time, the other parameters remaining at their reference value. The exact cost (180 s limit) is reported on class C and the ALNS cost (30 s, one seed) on the three classes (Table 6).

**Table 6: Sensitivity analysis around the reference instance ( $n = 15$ , C, medium windows,  $E = 15$  km,  $\sigma = 2$ ). The studied factor varies one at a time; the middle column corresponds to the reference (the endurance and speed factors equal their reference value). Exact cost (180 s limit) on C and ALNS cost (30 s, one seed) on the three classes.**

|           | Endurance (km) |         |         | Speed ratio |         |         | Windows |         |        |
|-----------|----------------|---------|---------|-------------|---------|---------|---------|---------|--------|
|           | 10             | 15      | 20      | 1.5         | 2.0     | 2.5     | tight   | medium  | wide   |
| C (exact) | 304.0          | 304.8   | 304.8   | 338.9       | 304.8   | 304.0   | 492.7   | 304.8   | 280.0  |
| C (ALNS)  | 314.95         | 314.95  | 314.95  | 314.95      | 314.95  | 314.95  | 708.72  | 314.95  | 288.16 |
| R (ALNS)  | 1243.12        | 1243.12 | 1176.94 | 1243.12     | 1243.12 | 1243.12 | 1403.52 | 1243.12 | 913.06 |
| RC (ALNS) | 1103.91        | 1004.84 | 1004.84 | 1005.01     | 1004.84 | 1004.74 | 1268.37 | 1004.84 | 843.26 |

The width of the time windows is the dominant factor. Switching to tight windows increases the ALNS cost by 125% on C ( $314.95 \rightarrow 708.72$ ), by 13% on R and by 26% on RC; wide windows reduce it by 8-27%. The endurance has a limited effect on the clustered class C, whose flights remain short (the cost is unchanged between 10 and 20 km), but a reduced endurance (10 km) penalizes RC (+10%) and an increased endurance (20 km) benefits R (-5%). The speed ratio has almost no effect on the classes under test: the flight times are short compared with the service

times, and the vehicle-drone synchronization (the wait at the recovery node) dominates the potential gain. Time windows therefore appear as the main cost lever, ahead of endurance and speed. The exact values of Table 6 are incumbents obtained with a still-large gap ( $\approx 69\%$  at  $n = 15$ ): the trends they indicate are consistent with the ALNS ones and should be read as indicative.

### Conclusion and Perspectives:-

This article proposes a unified MILP model and an ALNS heuristic for the mixed vehicle–drone routing problem with soft time windows. The compact formulation, based on ternary flight variables and an exact linearization of the synchronization, proves the optimality of the whole set of small instances ( $n = 10, 9/9$ ) and provides, on all the others, a feasible solution together with a lower bound; the ALNS efficiently handles the large instances. The experiments carried out on a reproducible Solomon-type benchmark show the economic contribution of the drones on every class of demand, the sensitivity of the cost to the time-window width, and the robustness of the ALNS, whose average gap to the best known stays below 1.2%. Several perspectives emerge, the introduction of several drones simultaneously in flight (multi-drone synchronization), the consideration of mid-tour recharging, the stochastic management of flight times (wind, air traffic), and the extension to heterogeneous vehicle fleets. Strengthening the lower bound for instance with combined flow-time inequalities generalizing the DFJ cuts is the most direct avenue to extend the scope of proven optimality beyond  $n = 10$ .

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