ISSN NO. 2320-5407

Journal homepage:http://www.journalijar.com
Journal DOI:10.21474/IJAR01

INTERNATIONAL JOURNAL OF ADVANCED RESEARCH

RESEARCH ARTICLE

Pseudo Finitely Quasi-Injectivesystems over monoids.

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Manuscript Info

Manuscript History:

Received: 19 March 2016
Final Accepted: 16 April 2016
Published Online: May 2016

Key words:
Pseudo finitely quasi-injective systems , Pseudo injective systems , Finitely quasi injective systems , Pseudo finitely injective systems.

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Abstract

The notion of pseudo injectivity relative to a class of finitely generated subsystems namely pseudo finitely quasi injective systems over monoidsis introduced and studied which is proper generalization of pseudo injective systems. Several properties of this kind of generalization as well as their characterizations are discussed. Conditions under which subsystems of pseudo finitely quasi injective system inherit this property. The relationship between the classes of pseudo finitely quasi injective with other classes of injectivity are studied.

Introduction:-

Throughout , S represents a monoid with zero element . A nonempty set M is called a unitary right S-system denoted by M_s , if there is a mapping f : M × S ⟶ M f(m,s) = ms such that : (1) m ·1=m (2) m(st) = (ms)t for all m ∈ M and s,t∈ S , where 1 is the identity element of S . Similarly we define a unitary left S-system . Throughout this work the basic S-system is a unitary right S-system . Let M_s , N_s be S-systems . A mapping $\alpha : M_s \rightarrow N_s$ is called S-homomorphism in case α(ms) = α(m)s for all s ∈ S and m ∈ M.

Let A_s , M_s be two S-systems . A_s is called M_s-injective if given an S- monomorphism $\alpha : N \rightarrow M_s$ where N is a subsystem of M_s and every S-homomorphism $\beta : N \rightarrow A_s$,can be extended to an S-homomorphism $\sigma : M_s \rightarrow A_s$ [7] . An S-system A_s is called injective if it is M_s-injective for all S-systems M_s . A_s is called quasi injective if it is A_s-injective .

An S-system M_s is called pseudo N_s- injective if each S-monomorphism from a subsystem of N_s into M_s extends to an S-homomorphism from $N_s$intoM_s. An S-system M_s is called pseudo injective if M_s is pseudo M_s-injective [8] .

In [5] , V.S.Ramamurthi introduced the concept of finitely injective modules . This concept motivate us to consider and study finitely injective systems relative to other S-systems as follows , an S-system M_s is called finitely N_s-injective (simply , F-N_s-injective) , if every homomorphism from a finitely generated subsystem of N_s to M_s extends to an S-homomorphism of N_s into M_s [6] . An S-system M_s is called finitely quasi injective(simply FQ-injective) if M_s is F-M_s-injective system .

A subset A of an S-system M_s is called a set of generating elements of $M_s$if every element m inM_s can be presented as m = as for some a ∈ A , s ∈S . ThusM_s is finitely generated if $M_s = < A >$ for some $|A| < \infty$, where $< A >$ is the subsystem of M_s generated by A([4], p.63) . An S-system N_s is called M_s-generated , where M_s is an S-system , if there exists an S-epimorphism $\alpha : M_s^{(I)} \longrightarrow N_s$ for some index set I . If I is finite , then N_s is called finitely M_s-generated of M_s [2].

In [2] , the authors introduced and studied pseudo-injective S-systems and obtained some results .In this work , we adopt generalizations of pseudo-injective and FQ-injective S-system .

Pseudo FinitelyQuasi Injective Systemsover Monoids:-

Definition(2.1): Let M_s and N_s be two S-systems . M_s is called pseudo finitely N_s-injective (simply PF-N-injective) if every monomorphism from a finitely generated subsystem of N_s into M_s extends to a homomorphism of N_s into M_s . An S-system M_s is called pseudo finitely quasi-injective (simplyPFQ-injective) if M_s is PF-M-injective system .A monoid S is called right PF-injective if S_s is pseudo FQ-injective .

Example and Remarks(2.2):-

(1) Every pseudo- injective(quasi-injective , injective)S-system is pseudo FQ-injective .Let S be the monoid {1,a,b,0} with $ab = a^2 = a$ and $ba= b^2 =b$, 0 is the zero element and 1 is the identity . S as a right S-system over itself is not pseudo FQ- injective, in fact consider the subsystemN={0,a,b} and α be S-monomorphismfrom N into S which defined by $\alpha(x) = \begin{Bmatrix} a & \text{if } x = b \\ b & \text{if } x = a \end{Bmatrix}$, and clearly α(0) = 0 .Then this S-monomorphism cannot be extended to S-endomorphism of S .

(2) The converse of (1) is not true in general , for example : let R with usual multiplication be R-system over itself . Then , take the basis $\{e_1,e_2,e_3,n_1,n_2,n_3,n_4\}$ of R with the following multiplication table :

	e_1	e_2	e_3	n_1	n_2	n_3	n_4
e_1	e_1	0	0	0	0	n_3	0
e_2	0	e_2	0	n_1	0	0	n_4
e_3	0	0	e_3	0	n_2	0	0
n_1	n_1	0	0	0	0	0	0
n_2	n_2	0	0	0	0	0	0
n_3	0	0	n_3	0	0	0	0
n_4	0	0	n_4	0	0	0	0

Then for R-system $M = e_2R$, the only five subsystems of M are(Θ) , $N_1 = n_1R$, $N_2 = n_4R$, $N_1 \oplus N_2 =(n_1 , n_2)R$ and M . It is easy to show that n_1R is not isomorphic to n_4R , therefore M is not quasi injective and any monomorphism from N_1 , N_2 or$N_1 \oplus N_2$ to M must be an inclusion map and hence can be lifted to identity map of M . This shows that M is pseudo injective (pseudo FQ-injective)

(3) It is clear that definition(2.1) is up to isomorphism . This means isomorphic system to pseudo FQ-injective is pseudo FQ-injective .Also , if M_s is pseudo F-N_1-injective with $N_1 \cong N_2$, then M_s is pseudo F-N_2-injective .

In the following theorem , we give characterizations of pseudo finitely quasi injective S-systems :

For an S-system M_s and fixed positive integers m and n . We write $M^{n\times m}$, for the set of all formal n×m matrices whose entries are elements in M . We will write also $M^n = M^{1\times n}$ and $M_n = M^{n\times 1}$.

Theorem(2.3) : The following statements are equivalent for an S-system M_s with $T = End_s(M_s)$:

(1) M_s is PFQ-injective .

(2) $\gamma_{S_n}(x)= \gamma_{S_n}(y)$, where $x , y \in M^n$, $n \in Z^+$ implies that Tx = Ty .

(3) If $x_i \in M_s$, i = 1, 2, … , n and $\alpha, \beta: \dot{\bigcup}_{i=1}^{n} x_iS \rightarrow M_s$ are monomorphism, then there exists S-homomorphism $\sigma \in T$ such that $\alpha = \sigma\beta$.

Proof : (1→2) Let $x,y \in M^n$ where $n \in Z^+$ and $x =(x_1 , x_2 , \dots , x_n)$, $y =(y_1 , y_2 , \dots , y_n)$. Define $\alpha: \dot{\bigcup}_{i=1}^{n} x_iS \rightarrow M_s$ by α(xs) = ys for each s ∈ S. If $xs = xs'$ for some s , $s' \in S_n$, then $(s,s') \in \gamma_{s_n}(x) \subseteq \gamma_{s_n}(y)$ which implies $ys = ys'$ and hence α is well-defined and it is clear that α is S-monomorphism. By (1) , there exists $\sigma \in T$ such that $\sigma$is an extension ofα . For each i = 1,2,…,n ,$y_i = \alpha(x_i) = \sigma(x_i)$, so $y = \sigma x$ and hence$Ty \subseteq Tx$. By similar argument ,we get $Tx \subseteq Ty$ and hence Tx = Ty .

(2→3) Since α , β are monomorphism , then $\gamma_{s_n}(\alpha(x)) = \gamma_{s_n}(\beta(x))$. By (2) , we have Tα(x) = Tβ(x) , for each $x \in M^n$. So , α(x) = σβ(x) for some σ ∈ T . Thus α = σβ .

(3→1) Take $\beta : \dot{\cup}_{i=1}^{n} x_iS \to M_s$ to be the inclusion mapping in (3) .

Corollary(2.4) : The following statements are equivalent for a monoid S :
(1) S is a right PF-injective .
(2) $\gamma_{S_n}(\alpha) = \gamma_{S_n}(\beta)$, where α , β ∈ S^n , n ∈ Z^+ implies that Sα = Sβ.
(3) If $a_i \in S$, i = 1, 2, … , n and $\alpha, \beta: \dot{\cup}_{i=1}^{n} a_iS \to S$ are monomorphism , then there exists S-homomorphism b ∈ S such that α = bβ .

In the following theorem we get another form of theorem(2.3) . First , let M_s be S-system . For all element $x = (x_1, \dots, x_n) \in M^n$ and α , β ∈ T = End(M_s) ,define the following three sets :

$A_x = \{ y \in M^n \mid \gamma_{s_n}(x) = \gamma_{s_n}(y)\}$;
$S_{(\alpha,x)} = \{ \beta \in T \mid \ker\beta \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) = \ker\alpha \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))\}$;
$B_x = \{\alpha \in T \mid \ker\alpha \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) = I_{x_iS}\}$. Where I_{x_iS} is the trivial congruence on x_iS for each i .

In fact A_x (respectively $S_{(\alpha,x)}$) is an equivalence class of the following equivalence relation on M^n . For x,y∈ M^n, x ~ y iff $\gamma_{s_n}(x) = \gamma_{s_n}(y)$ and for x ∈ M^n , α , β ∈ T , we say α ≈ β if and only if
$\ker\alpha \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) = \ker\beta \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))$.

Theorem(2.5) : Let M_s be an S-system with T=End(M_s) , the following conditions are equivalent:
(1) M_s is PFQ-injective ,
(2) $A_x = B_x x$, for all x in M^n ,
(3) If $A_x = A_y$, then $B_x x = B_y y$,
(4) For every S-monomorphism $\alpha, \beta: \dot{\cup}_{i=1}^{n} x_iS \to M_s$, there exists S-homomorphism σ ∈ T such that α = σβ .

Proof : (1→2) Let y = (y_1 , … , y_n) ∈ A_x , this implies $A_x = A_y$, $\alpha: \dot{\cup}_{i=1}^{n} x_iS \to M_s$ is defined by α(xs) = ys . It is obvious that α is well-defined and S-monomorphism . Since M_s is PFQ-injective , so by (1) , there exists σ ∈ T such that σ extends α , then y= α(x) = σ(x) , where i = 1 , 2 ,… , n , so y = σx . This means that , ∀ x = (x_1 , … , x_n) ∈ M^n , we have y = α(x) = σ(x) = σ • x , so σ ∈ B_x (In fact , if (xs , xt) ∈ $\ker\sigma \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))$,then σ(xs) = σ(xt) and xs = xt . So , $\ker\sigma \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) = I_{x_iS}$) . Thus , $A_x \subseteq B_x x$. Conversely , if σ x ∈ $B_x x$, then σ ∈ B_x , that is $\ker\sigma \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) = I_{x_iS}$. It is obvious that $\gamma_{s_n}(x) \subseteq \gamma_{s_n}(\sigma x)$, since for (r, s) ∈ $\gamma_{s_n}(x)$, we have xr = xs , since σ is well-defined , so σ(xr) = σ(xs) . Thus , σ(x)r = σ(x)s which implies that (r,s) ∈ $\gamma_{s_n}(\sigma x)$. Now, if σ(xr) = σ(xs) and (xr , xs) ∈ $\ker\sigma \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) = I_{x_iS}$, then xr =xs and (r,s) ∈ $\gamma_{s_n}(x)$. Hence , $\gamma_{s_n}(\sigma x) \subseteq \gamma_{s_n}(x)$. Then , $\gamma_{s_n}(\sigma x) = \gamma_{s_n}(x)$. Therefore , σx ∈ A_x and $B_x x \subseteq A_x$.

(2→3) Let $A_x = A_y$. Then , $A_x = B_x x$, $A_y = B_y y$. So , $B_x x = B_y y$.

(3→4) Let $\alpha : \dot{\cup}_{i=1}^{n} x_iS \to M_s$ and $\beta : \dot{\cup}_{i=1}^{n} x_iS \to M_s$ be S-monomorphisms . Then , for x = (x_1 , … , x_n) , $\gamma_{s_n}(\beta x) = \gamma_{s_n}(\alpha x)$. Since , for (s, t) ∈ $\gamma_{s_n}(\beta x)$, then β(xs) = β(xt) . Since β is monomorphism , so xs = xt . Since α is well-defined , so α(xs) = α(xt) . This means $\gamma_{s_n}(\beta x) \subseteq \gamma_{s_n}(\alpha x)$. In similar way we can prove $\gamma_{s_n}(\alpha x) \subseteq \gamma_{s_n}(\beta x)$, which implies $A_{\alpha x} = A_{\beta x}$, then by(3) $B_{\alpha x}\alpha x = B_{\beta x}\beta x$. Since $\ker I_M \cap (\alpha(xS) \times \alpha(xS)) = I_{\alpha(xS)}$, so $1_M \in B_{\alpha x}$. Then αx ∈ $B_{\beta x}\beta x$, so there exists σ ∈ $B_{\beta x}$ such that α = σβ .

(4→1)Let β = I_{x_iS} be the inclusion map of $\dot{\cup}_{i=1}^{n} x_iS$ in(4), so we obtain the required .

Proposition(2.6): Let M_s be PFQ-injective S-system with T = End(M_s) . Then , for α ∈ T , we have :

$S_{(\alpha,x)} = B_{\alpha x}\alpha \cup \ell_T(x_iS \times x_iS)$, $\forall x \in M^n$

Proof : Let $\beta \in S_{(\alpha,x)}$. Then , $\ker\beta \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) = \ker\alpha \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))$. We claim that $\gamma_{s_n}(\alpha x) = \gamma_{s_n}(\beta x)$. In fact , if $(s,t) \in \gamma_{s_n}(\alpha x)$, then $\alpha(xs) = \alpha(xt)$ which implies $(xs,xt) \in \ker\alpha \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))$ and $(xs,xt) \in \ker\beta \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))$ which implies $\beta(xs) = \beta(xt)$ and then $\beta(x)s = \beta(x)t$. Thus $(s,t) \in \gamma_{s_n}(\beta x)$. Hence , $\gamma_{s_n}(\alpha x) \subseteq \gamma_{s_n}(\beta x)$, similarly we have $\gamma_{s_n}(\beta x) \subseteq \gamma_{s_n}(\alpha x)$ and then we obtain $\gamma_{s_n}(\alpha x) = \gamma_{s_n}(\beta x)$. Then , we have $\beta \in A_{\alpha x}$. Since $A_{\alpha x} \subseteq B_{\alpha x}\alpha x$, by theorem (2.5) , so $\beta \in B_{\alpha x}\alpha x$ and since $\beta(xs) = \beta(xt)$, where $\beta \in T$, thus $\beta \in \ell_T(x_iS \times x_iS)$and then $\beta \in B_{\alpha x}\alpha \cup \ell_T(x_iS \times x_iS)$. This means $S_{(\alpha,x)} \subseteq B_{\alpha x}\alpha \cup \ell_T(x_iS \times x_iS)$ …(1) . Conversely , let $\beta \in B_{\alpha x}\alpha \cup \ell_T(x_iS \times x_iS)$. If $\beta \in \ell_T(x_iS \times x_iS)$, so $\beta \in T$ and $\beta(x_is) = \beta(x_it)$. If $\beta \in B_\alpha\alpha$, so there exists $\varphi \in B_\alpha$ such that $\beta = \varphi o \alpha$. Also , $\ker\varphi \cap (\dot{\cup}_{i=1}^{n}(\alpha(x_iS) \times \alpha(x_iS))) = I_{\alpha(x_iS)}$ and $\ker\beta \cap (\dot{\cup}_{i=1}^{n}(\alpha(x_iS) \times \alpha(x_iS))) = I_{\alpha(x_iS)}$. Now, if $(xs,xt) \in \ker\varphi\alpha \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))$, then $\varphi\alpha(xs) = \varphi\alpha(xt)$. Hence $(\alpha(xs),\alpha(xt)) \in \ker\varphi \cap (\dot{\cup}_{i=1}^{n}(\alpha(x_iS) \times \alpha(x_iS))) = I_\alpha$. This implies that $(xs,xt) \in \ker\alpha \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))$ .Thus , $\ker\beta \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) \subseteq \ker\alpha \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))$ …(1) . If $(xs,xt) \in \ker\alpha \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))$, so $\alpha(xs) = \alpha(xt)$, since $\varphi \in T$, so $\varphi\alpha(xs) = \varphi\alpha(xt)$ which implies $\beta(xs) = \beta(xt)$ and then $(xs,xt) \in \ker\beta \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))$. Thus , $\ker\alpha \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) \subseteq \ker\beta \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))$ …(2) . From (1) and (2) , we have $\ker\alpha \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) = \ker\beta \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))$ and then $\beta \in S_{(\alpha,x)}$.

Proposition(2.7) : Let M_s be PFQ-injective S-system with T = End (M_s) and $\alpha \in T$, $x \in M^n$. Then :$\alpha \in B_x$ if and only if $B_x = B_{\alpha x}\alpha \cup \ell_T(x_iS \times x_iS)$.

Proof : ⇒) Let $\alpha \in B_x$ and $f \in S_{(\alpha,x)}$, so $\ker f \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) = \ker\alpha \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))$, but $\ker\alpha \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) = I_{x_iS}$, hence $\ker f \cap ((\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) = I_{x_iS}$, which implies $f \in B_x$. Thus , $S_{(\alpha,x)} = B_x$, so by proposition (2.6)$B_x = B_{\alpha x}\alpha \cup \ell_T(x_iS \times x_iS)$.

⇐) Assume that $B_x = B_{\alpha x}\alpha \cup \ell_T(x_iS \times x_iS)$ and $\alpha \in T$, $\alpha \notin B_x$. Then , we have $\ker\alpha \cap ((\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) \neq I_{x_iS}$, so there exists $(xs,xt) \in \ker\alpha \cap ((\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))$ with $xs \neq xt$, then $\alpha(xs) = \alpha(xt)$. Since$1_M \in B_m$, so $\ker I_M \cap ((\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) = I_{x_iS}$. But , since $S_{(\alpha,x)} = B_x = B_{\alpha x}\alpha \cup \ell_T(x_iS \times x_iS)$, hence $I_M \in S_{(\alpha,x)}$, and then $\ker\alpha \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) = \ker I_M \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))$. Thus , $\ker\alpha \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS) = I_{x_iS}$ which implies xs = xt and this is a contradiction with xs ≠ xt . This means that $\alpha \notin B_x$ implies a contradiction . Thus , $\alpha \in B_x$.

Proposition(2.8): Let $M_s$be a PFQ-injective S-system with T =End(M_s) and $S_{(\alpha,x)} = B_{\alpha x}\alpha \cup \ell_T(x_iS \times x_iS)$ for all $\alpha \in T$ and all $x \in M^n$. If $A_{\alpha x} = A_{\beta x}$, then $\beta \in B_{\alpha x}\alpha \cup \ell_T(x_iS \times x_iS)$.

Proof : Let $A_{\alpha x} = A_{\beta x}$, then $\gamma_{s_n}(\alpha x) = \gamma_{s_n}(\beta x)$. Let (xs,xt) ∈ kerα , so α(xs) = α(xt) where $x \in M^n$ and $s,t \in S_n$. Then , α(x)s = α(x)t , so $(s,t) \in \gamma_{s_n}(\alpha(x)) = \gamma_{s_n}(\beta(x))$. This implies β(x)s= β(x)t and then β(xs) = β(xt) , this means (xs,xt) ∈ kerβ . Thus kerα ⊆ kerβ . Similarly for the other direction. Thus, kerα = kerβ . So , $\ker\beta \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS)) = \ker\alpha \cap (\dot{\cup}_{i=1}^{n}(x_iS \times x_iS))$ which implies $S_{(\alpha,x)} = S_{(\beta,x)}$, so by hypothesis , we have $B_{\alpha x}\alpha \cup \ell_T(x_iS \times x_iS) = B_{\beta x}\beta \cup \ell_T(x_iS \times x_iS)$. Since $1_M \in B_{\beta(x)}$. This means $\beta = 1_M \bullet \beta \in B_{\beta x}\beta$, so $\beta \in B_{\beta x}\beta \cup \ell_T(x_iS \times x_iS) = B_{\alpha x}\alpha \cup \ell_T(x_iS \times x_iS)$, this implies $\beta \in B_{\alpha x}\alpha \cup \ell_T(x_iS \times x_iS)$.

The following proposition gives a condition under which subsystem of PFQ-injective inherit this property . Before this , we need the following concept :

Recall that a subsystem N of S-system M_s is fully invariant of M_s if f(N) ⊆ N , for all f ∈ $End_s(M_s)$ [3] . An S-system is called duo if each subsystem of it is fully invariant .

Proposition(2.9) : Every fully invariant subsystem of PFQ-injective system is PFQ-injective .

Proof : Let M_s be PFQ-injective system and N be a fully invariant subsystem of M_s . Let X be any finitely generated subsystem of N and f be S-monomorphism from X into N . Since M_s is PFQ-injective system , so there exists an S-endomorphism g of M_s such that $goi_Noi_X = i_Nof$, where i_X and i_N are the inclusion maps of X into N and N into M_s respectively . As N is fully invariant in M_s , so $g(N) \subseteq N$. Put $g_{|N} = h$, then , $\forall x \in X$, we have $(hoi_X)(x) = g(x) = (goi_Noi_X)(x) = (i_Nof)(x) = f(x)$. Therefore N is PFQ-injective system .

Recall that an S-system M_s is called multiplication if every subsystem of M_s is of the form MI for some right ideal I of S . It is clear that every subsystem of multiplication system is fully invariant [3] .

Corollary(2.10) : If M_s is PFQ-injective duo (multiplication) S-system , then every subsystem of M_s is PFQ-injective .

Proposition(2.11) : Let $M_s$and N_s be two S-systems and $N^/$ a subsystem of N_s . If M_s is PFN_s-injective (respectively FN_s-injective) , then :

(1) Every retract of M_s is PFN-injective (respectively FN-injective) .
(2) M_s is PF$N^/$-injective (respectively F$N^/$-injective).

Proof :(1) Let $M_s = M_1 \oplus M_2$,and K befinitely generated subsystem ofN and f be S-monomorphism (resp. homomorphism) ofK into M_1 . Since M_s is PF-N_s-injective (resp. FN_s-injective) , so ($j_1$of) where j_1 is injection of M_1 into M_s extends to S-homomorphism g of N_s into M_s such that $goi_K = j_1of$. Put $g^/ (= \pi_1 g) : N_s \to M_1$, where $\pi_1$be the projection map of M_s into M_1 , then $g^/oi_K = \pi_1 ogoi_K = \pi_1 oj_1of = I_{M_1}of = f$. Thus f extends to S-homomorphism $g^/$ and M_1 is PF-N_s-injective system .

(2) It is obvious .

The following corollaries is immediately from above proposition :

Corollary(2.12): Retract of PFQ-injective system is PFQ-injective .

Corollary(2.13) : Let N be any subsystem of S-system M_s . If N is PF-M_s-injective , then N is pseudo finitely injective .

Proposition(2.14) : Let $M_s = M_1 \oplus M_2$be the direct sum of subsystems M_1 , M_2 . If M_2 is PF-M_1-injective , then for each finitely generated subsystem N of M_s with $N \cap M_1 = \Theta$, $N \cap M_2 = \Theta$, there exists a subsystem $M^/$ of M_s such that $M_s = M^/ \oplus M_2$and N is subsystem of $M^/$.

Proof : Let $\pi_i : M_s \to M_i$, where i = 1,2 denoted the projection mapping and $\alpha = \pi_{1|N}$, $\beta = \pi_{2|N}$. Then , α and β are two S-monomorphisms . By assumption , there exists an S-homomorphism $\varphi : M_1 \to M_2$ such that $\varphi o \alpha = \beta$. Let $M^/ = \{ (x, \varphi(x)) \mid x \in M_1\}$. It is easy to check that $M_s = M^/ \oplus M_2$ and N is a subsystem of $M^/$.

Proposition(2.15) : Let $M_s$and N_s be two S-systems . Let $N_s$be finitely generated subsystem of S-system M_s . Then $N_s$is PF-M_s-injective if and only if every monomorphism $f : N_s \to M_s$ split .

Proof : Assume that N_s is PF-M_s-injective system and $f : N_s \to M_s$ be monomorphism , then by PF-M_s-injective of N_s , there exists an S-homomorphism $g : M_s \to N_s$ such that $gof = I_N$. Since $N_s \cong f(N_s)$, so $f(N_s)$ is a retract of M_s. Conversely , assume that A is finitely generated subsystem of M_s . Then , by assumption the monomorphism (inclusion map) i_A of A into M_s split , this means there exists $\omega : M_s \to A$ such that $\omega oi_A = I_A$. Now , for S-monomorphism $f : A \to N_s$, set set $g (= fo\omega) : M_s \to N_s$ which implies that $goi_A = f o\omega oi_A = f oI_A = f$. Thus N_s is PF-M-injective system .

Corollary(2.16) : Let N_s be a finitely generated subsystem of an S-system M_s . If N_s is PF-M_s-injective system , then N_s is a retract of M_s .

Corollary(2.17) : Let M_s be PFQ-injective S-system . Then , every finitely generated subsystem of M_s which is isomorphic to M_s is a retract of M_s .

Definition(2.18) : An S-system M_s is called FC_2 if every finitely generated subsystem of M_s that is isomorphic to a retract of $M_s$is itself a retract of M_s .

Theorem(2.19) : Every PFQ-injective system satisfies FC_2 .

Proof : Let M_s be PFQ-injective S-system and A be a retract of M_s with $A \cong B$, where B is finitely generated subsystem ofM_s . Let f be S-isomorphism from B into A, then f is S-monomorphism from B into M_s . Since A is a retract of M_s, so by proposition (2.11)(1) A is PF-M_s-injective system . By example and remarks (2-2)(2) , since $A \cong B$, so B is PF-M_s-injective system . Then , by proposition (2.15) f is split and by corollary (2.16) B is a retract of M_s and so M_s satisfies FC_2 – condition .

Proposition(2.20) : Let M_s be an S-system and $\{N_i\}_{i\in I}$ be a family of S-systems , where I is finite index set . Then ,$\Pi_{i\in I}N_i$ is pseudo finitely M-injective if and only if for each $i \in I$, N_i is pseudo finitely M-injective system .

Proof:⇒)Put $N_s = \Pi_{i\in I}N_i$, assume that N_s is PF-M-injective S-system and A is a finitely generated subsystem of M_s. Let f be an S-monomorphism of A into N_i. Since N is PF-M_s-injective , so there exists S-homomorphism $g : M_s \rightarrow N_s$ such that $goi_A = j_i of$, where j_i is the injection map of N_i into N_s and i_A is the inclusion map of A into M_s . Now , let π_i be the projection map of N onto N_i . Put $h(=\pi_i og)$: $M_s \rightarrow N_i$, then $\forall a \in A$, $(hoi_A)(a) = (\pi_i ogoi_A)(a) = (\pi_i oj_i of)(a) = f(a)$. Thus N_i is PF-M-injective system.

⇐) Assume that N_i is PF-M_s-injective for each $i \in I$. Let A be finitely generated subsystem of M_s and f be an S-monomorphism of A into N_s . Since N_i is PF-M_s-injective S-system, so there exists S-homomorphism $\beta_i : M_s \rightarrow N_i$such that $\beta_i oi_A = \pi_i of$, where i_A be the inclusion map of A into M_s . Now, define an S-homomorphism β $(=j_i o\beta_i)$: $M_s \rightarrow N_s$, then $\beta oi_A = j_i o\beta_i oi_A = j_i \, o\pi_i of = f$. Therefore, N_s is PFM_s-injective system .

Corollary (2.21) : Let M_s and N_i be S-systems , where $i \in I$ and I is finite index set . If $\oplus_{i\in I}N_i$ is PF-M_s-injective for all $i \in I$, then N_i is PF-M_s-injective .

Proposition(2.22) : If M_s is pseudo finitely injective S-system and $T = End(M_s)$, then TA = TB for each isomorphic subsystems A and B of M_s .

Proof : By assumption there exists an S-isomorphism $\alpha : A \rightarrow B$, let $b \in B$ so there exists $a \in A$ such that $\alpha(a) = b$. For $s,t\in S$, if as = at , so bs = bt , which implies that $\gamma_s(a) \subseteq \gamma_s(b)$. Since $M_s$is pseudo finitely injective (or PFQ-injective) , then by theorem (2.3) , $Tb \subseteq Ta$ and hence $Tb \subseteq TA$ $\forall$ $b \in B$. Thus $TB \subseteq TA$. Similarly , we can prove $TA \subseteq TB$. Therefore TA = TB .

As an immediate consequence of above proposition , we have the following result :

Corollary(2.23): If S is pseudo finitely injective monoid and A , B are two isomorphic ideal of S , then A = B .

Recall that two S-systems M_s and N_s are mutually finitely injective (respectively PF-injective) if M_s is finitely N_s-injective (respectively PF-N_s-injective) and N_s is finitely M-injective (resp. PF-M_s-injective) [6] .

Theorem(2.24) : If $M_1\oplus M_2$ is PFQ-injective system , then M_1 and M_2 are mutually F-injective system . In particular , if M_s is S-system such that $M \oplus M$ is PFQ-injective , then M_s is FQ-injective .

Proof : Let $M_1\oplus M_2$ be PFQ-injective system . Let X be any finitely generated subsystem of M_2 and f be S-homomorphism from X into M_1 . Define $\alpha:X \rightarrow M_1\oplus M_2$ by $\alpha(x) = (f(x),x)$, $\forall$ $x \in X$, then it is clear that α is monomorphism (in fact for $\alpha(x_1) = \alpha(x_2)$, then we have $(f(x_1),x_1) = (f(x_2),x_2)$, so $f(x_1) = f(x_2)$ with $x_1 = x_2$) . By proposition (2.11)(2) , $M_1\oplus M_2$ is PF-M_2-injective , so α extends to S-homomorphism $g : M_2 \rightarrow M_1\oplus M_2$. If π_1 : $M_1\oplus M_2 \rightarrow M_1$ is the natural projection , then $h(=\pi_1 g)$: $M_2 \rightarrow M_1$ is S-homomorphism extending f . Consequently , M_1 is F-M_2-injective system .

The proof of the following corollary is immediately from above theorem and proposition (2.11) :

Corollary(2.25): If $\oplus_{i\in I}M_i$ is PFQ-injective system , then M_j is F-M_K-injective for all distinct j , k ∈ I .

Relation among Pseudo FQ-Injective S-systems with other Classes of Injectivity :

The following proposition explain under which condition on pseudo finitely N_s-injective for each S-system $N_s$to be injective :

Proposition(3.1) : Let M_s be a finitely generated S-system . Then M_s is injective system if and only if M_s is pseudo finitely N_s-injective for each S-system N_s.

Proof : ⇒) It is obvious .

⇐) Let $E = E(M_s)$be the injective hull of S-system M_s . Then M_s is a finitely generated subsystem in $M_s \oplus E(M_s)$. Let $i : M_s \rightarrow E(M_s)$ be the inclusion mapping , $j : E(M_s) \rightarrow M_s \oplus E(M_s)$ the natural injection and $I_M : M_s \rightarrow M_s$ the identity mapping . Since M_s is PF-$M_s \oplus E(M_s)$ – injective , so this implies that I_M can be extended to S-homomorphism $f : M_s \oplus E(M_s) \rightarrow M_s$ .This meansM_s is a retract of $E(M_s)$ and since $E(M_s)$ is injective , so M_s is injective .

As a particular case of above proposition , we have the following corollary :

Corollary(3.2) : A monoid S is self-injective if and only if S is pseudo finitely S-injective S-system .

The following proposition explain under which condition on pseudo finitely quasi injective to be injective , but before this we need the following concept :

Definition(3.3) : An S-system M_s is said to be weakly injective if for every finitely generated subsystem N of $E(M_s)$, we have $N \subseteq X \subseteq E(M_s)$ for some $X \cong M_s$.

Proposition(3.4) : Let M_s be a finitely generated system . Then M_s is injective system if and only if M_s is weakly injective and PFQ-injective .

Proof : ⟹)It is obvious .

⟸) It is enough to prove that $M_s = E(M_s)$. Let $x \in E(M_s)$, so $M_s \cup xS$ is finitely generated . As M_s is weakly injective , so there exists subsystem X of $E(M_s)$ such that $M_s \cup xS \subseteq X \cong M_s$. Since M_s is PFQ-injective system , so X is also PFQ-injective by Example and Remarks (2.2)(2) . By theorem(2.19) X is satisfy FC_2 and since M_s is finitely generated subsystem of X , soM_s is a retract of X . But M_s is ∩-large subsystem of $E(M_s)$, so M_s is ∩-large in X . Therefore $M_s = X$, and $x \in M_s$.

It is clear that every finitely quasi injective system (FQ-injective) is pseudo finitely quasi injective system (PFQ-injective) , but the converse is not true in general , the following proposition give under which condition for PFQ-injective system to being FQ-injective , but we need the following concept and theorem :

Recall that a congruence ρon an S-system M_s is called large congruence , if for every congruence α on M_s with $\alpha \neq I_M$ (the trivial congruence) , we have $\alpha \cap \rho \neq I_M$ [2]. Then , an S-system M_s is called cog-reversible if each congruence ρ on $M_s$with $\rho \neq I_M$ is large on M_s, where I_M is the trivial congruence on M_s[2] .

Theorem(3.5) [2]**:**Let M_s be a cog-reversible nonsingular S-system with $\ell_M(s) = \Theta$ for each $s \in S$ .Then M_s is pseudo injective system if and only if M_s is quasi injective .

Proposition(3.6) : Let M_s be a cog-reversible nonsingular S-system with $\ell_M(s) = \Theta$ for each $s \in S$. Then M_s is FQ-injective system if and only if M_s is PFQ-injective.

Proof : Assume that an S-system M_s is PFQ-injective . Let N be finitely generated subsystem of M_s and f be S-homomorphism from N into M_s. If f is S-monomorphism, then there is nothing to prove . Let f is not S-monomorphism , then by the proof of theorem(3.5) , we get that f is zero map . Then , M_s is FQ-injective system .

The following proposition give a condition for PFQ-injective system to bepseudo injective , but we need the following concept :

Recall that an S-system M_s is Noetherian if every subsystem of M_s is finitely generated . A monoid S is right Noetherian if S_s is Noetherian . Equivalently , S is right Noetherian if and only if S satisfies the ascending chain condition for right ideals.

The proof of the following proposition is immediately :

Proposition(3.7) : Let M_s be Noetherian S-system . Then M_s is pseudo injective system if and only if M_s is PFQ-injective .

Recall that an S-system A_s is called regular acts if and only if for any $a \in A_s$ the cyclic subsystem (S-cyclic) is projective(corollary19.3) [4 , p.301] .

Definition(3.8) : An S-system A_s is called pseudo regular if every finitely generated subsystem of A_s is a retract of A_s .

The following theorem is a generalization of theorem(10) in [8] and the proof is immediately by theorem(2.19) :

Theorem(3.9) : An S-system M_s is pseudo regular if and only if M_s is PFQ-injective and every finitely generated subsystem of M_s isomorphic to a retract of M_s .

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