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RESEARCH ARTICLE

A NEW APPROACH TOWARDS TRANSFORMATION OF BOTH CARDINAL AND ORDINAL 2×2 GAMES.

M. Jawad sajid[1] Qingren Cao[1] Li Xinchun[1] and M. Fahad sajid[2].
1. School of management, China University of mining and technology, Xuzhou, Jiangsu province, 221116, China.
2. Department of management sciences, COMSATS institute of information technology, Islamabad, Pakistan.

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Abstract

Game theory literature largely lacks a generally accepted method, capable of transforming both cardinal and ordinal 2×2 games or games where one player is cardinal and the other one is ordinal. We devised a new method to overcome this problem. We used propositional logic to represent our general argument. Afterwards developed basic inferences, conclusion is derived according to modus ponens, based on these inferences we derived biconditional statement and truth table. By adding and subtracting same figure to/from opposite expected outcomes (according to change in preference of one or both players) games having ordinal and/or cardinal payoffs can be transformed. We used the Notorious "Prisoners Dilemma" as an example and transformed it under our new method. Our method will not only expand the available knowledge in game theory by including both ordinal and cardinal transformations but it can also be helpful for the potential development of new taxonomies and topologies.

Introduction:-

2×2 Games probably are the simplest and the most important games represented in strategic form. They are the first to be thought and last to be forgotten *[1]*. 2×2 Games are defined as having two players with two alternatives facing four possible consequences *[2]*. Models related to conflicts have been extensively demonstrated using 2×2 games see *[3]*; *[4]*; *[5]*; *[6]*; *[7]*; *[8]*.

Rapoport and Guyer *[9]* classified the strict ordinal 2×2 games where each agent has strict preference towards four possible consequences. Kilgour and Fraser *[10]* developed "Taxonomy" for 726 games where one or both agents can possibly have similar preferences towards one or more consequences ("weak preferences allowing ties"). Fishburn and Kilgour *[11]* calculated "Strategically distinct binary 2×2 games" on basis of "dominant strategies and/or pure strategy Nash equilibrium", where agents' preferences are not bounded by "Transitivity requirements". A much more comprehensive work related to transformation of 2×2 games has been done by Robinson and Goforth *[1]*. The basic approach for transformation adopted by them is: transformation through swaps. Number of swaps determines the closeness between original and newly transformed game. For more detailed comprehension of the notion, see, for example, *[12] [13]*; *[14]*; *[15]*; *[16]*; *[17]*.

Corresponding Author:- M. Jawad sajid.
Address:- School of management, china university of mining and technology, xuzhou, jiangsu province, 221116, china.

Although there is work involving both ordinal and cardinal games *[18]*; *[19]*; *[20]*; *[21]* but there is still no generally accepted method for the transformation of both cardinal and ordinal games. In spite of the fact that both cardinal and ordinal payoffs are integral part of game theory, most of the available literature in game theory mainly focuses on ordinal transformations alone. However there might be a situation where both cardinal and ordinal transformations are required. Or a situation may arise where only cardinal payoffs can be considered for example "Mixed Strategy Games". So there is an immense need for a method that is capable of taking in to account both cardinal and ordinal payoffs.

In this paper we have presented a simple and effective method for transformation of both ordinal and cardinal 2×2 games. We have argued that an increased preference towards a particular strategy should result in a decreased preference towards the opposite strategy. Following is the organization of our paper. In Section 1 we have presented Notions and preliminaries. Section 2 contains general argument, its logical representation, and derivation of biconditional statement and development of truth table based on biconditional statement. Section 3 is the "Example" illustrates the practical implementation of our proposed method. In Section 4 we have discussed limitations and implications of our work.

Notions and Preliminaries:-

In this section we have provided a brief introduction of the terminologies used in our paper.

Let:

$i = Set\ of\ Players$

$$i = \{1,2\} \quad \textbf{(1)}$$

$S = Set\ of\ Strategies$

$$S = \{s_1, s_2\} \quad \textbf{(2)}$$

$A = \{Set\ of\ values\ representing\ change\ in\ preference\}$

$$A = \{a_{1,\cdots,}a_n\} \quad \textbf{(3)}$$

Player 1,s Payoff as a function of the vector of strategies adopted is presented by function:

$u_1: S \to R$ Where $u_1(s_1)$ is the payoff available to player 1 for playing strategy s_1

And $u_1(s_2)$ is the payoff available for player 1 for playing strategy s_2.

Player 2,s payoff as a function of strategies adopted is presented by function:

$u_2: S \to R$ Where $u_2(s_1)$ is the payoff available to player 2 for playing Strategy s_1.

And $u_2(s_2)$ is the payoff available to player 1 for playing Strategy s_2.

General argument Logical representation:-

We have argued that an increased preference towards one strategy should result in a decreased preference towards the opposite strategy. In the first section we have developed basic inferences based on propositional logic and conclusion is derived according to Modus ponens[1]. From these we have obtained our biconditional statement and afterwards we have developed truth table for bicondiotional statement.

Development of premises and derivation of conclusions:-

Premise 1: If preference towards s_1 increases then preference towards s_2 decreases.

Premise 2: Preference towards s_1 increases

Conclusion: Preference towards s_2 decreases

The conclusion and the two premises are propositions. By using the function of "Modus ponens (an inference rule) we can have the conclusion while both premises are axiomatic.

These minuscule statements can be replaced by statement letters (variables).

Premise 1: $P(S_1) \to Q(S_2)$ **(4)**

Premise 2: $P(S_1)$ **(5)**

Conclusion: $Q(S_2)$ **(6)**

[1] "If you know that both P and P→Q are true you can conclude that Q is also true" [22].

It also implies that
Premise 1: If preference towards s_2 decreases then preference towards s_1 increases.
Premise 2: Preference towards s_2 decreases
Conclusion: Preference towards s_1 increases

Premise 1: $Q(S_2) \rightarrow P(S_1)$ **(7)**

Premise 2: $Q(S_2)$ **(8)**

Conclusion: $P(S_1)$ **(9)**

Biconditional statement:-
From the above mentioned two conditions we can easily obtain a bicondotional statement.

$$\frac{P(S_1) \rightarrow Q(S_2), Q(S_2) \rightarrow P(S_1)}{P(S_1) \leftrightarrow Q(S_2)} \quad \textbf{(10)}$$

Now the rule in whatsoever cases "$P(s_1) \rightarrow Q(s_2)$" and "$Q(s_2) \rightarrow P(s_1)$" presented on line of a proof can legitimately be placed on a consequent line.
More formally (succinctly) we can write the above statement as

$$(P(S_1) \rightarrow Q(S_2)), (Q(S_2) \rightarrow P(S_1)) \vdash (P(S_1) \leftrightarrow Q(S_2)) \quad \textbf{(11)}$$

Here $P(s_1)$ represents an increase in preference towards strategy $(s_1)$and $Q(s_2)$ represents a decrease of preference towards strategy(s_2) . Our symbolic expressions are the exact representation of the above mentioned expressions in natural language.

Also is the case these expressions can additionally respond to any other this type of inference, validated on the same foundation. Where $\vdash$ is a Metalogical symbol[2] meaning that $P(s_1) \leftrightarrow P(s_2)$ are in a syntactic consequence[3] when $P(s_1) \rightarrow P(s_2)$ and $P(s_2) \rightarrow P(s_1)$ are both in a proof

Table 1:- Truth table for biconditional statement

$\mathbf{P(s_1)}$	$\mathbf{Q(s_2)}$	$\mathbf{P(s_1) \leftrightarrow Q(s_2)}$
T	*T*	*T*
T	*F*	*F*
F	*T*	*F*
F	*F*	*T*

We can see that $P(s_1) \leftrightarrow Q(s_2)$ is true only when both $P(s_1)$ and $Q(s_2)$ are either “True” or “False”.

Transformation example:-
Ordinal transformation:-
For ordinal transformations simply add the difference between the opposite payoffs to the outcome whose preference has been increased and at the same time subtract the same amount from the opposite outcome, which will in turn transform one ordinal 2×2 game in to another (in same manner as achieved through swapping), see *[1]*.One thing should be kept in mind player 1 payoffs are presented on vertical axis and player 2 on horizontal axis.

	Corporate	*Defect*
Corporate	*3,3*	*1,4*
Defect	*4,1*	*2,2*

Figure 1:- Classical prisoners’ dilemma
Let’s suppose that player’s one preference towards expected outcome (C, C) has been increased resulting in a decreased preference towards opposite outcome (D, C), While player two preferences are constant. As the difference

2 It says that the statements on each side of it are fundamentally the same in spite of the value of the components [23]
3 *“A sentence j is a syntactic consequence of a set of sentences K in propositional logic if there is a proof with premises K and conclusion j”* [24].

between Cooperate and Defect in absolute value terms ($C - D = 1$) i.e. ($3 - 4 = 1$), so as per our proposed definition we should add 1 to cooperate payoff and deduct the same from Defect's payoff (highlighted in the diagram).

	Corporate	*Defect*
Corporate	*3+1,3*	*1,4*
Defect	*4-1,1*	*2,2*

Figure 1.2:- Under Operation

	Corporate	*Defect*
Corporate	*4,4*	*2,2*
Defect	*3,1*	*4,3*

Figure 2:- Newly transformed Asym dilemma.

For our example we have taken smallest possible change in preference and transformed prisoners' dilemma in to another type of dilemma i.e. Asym dilemma. This name of the game has been taken from *[12]*, Appendix 1 contains eight conditions for possible transformation of 2×2 games .Because of the fact that transformed game is also another kind of dilemma so Nash equilibrium remains the same red color indicates Nash equilibrium. For objectivity purpose we have considered only one game type, preference change of one player towards one expected outcome and transformed the game only one time.

Cardinal transformations:-

For cardinal transformations we have to add and deduct the change in preference in dollar units', quantity, market share, utility, etc. i.e. the degree or intensity of change in preference will be considered. It could be any amount ($A = \{a_1, \dots, a_n\}$) depending upon the intensity of change in preference. In this paper we will not discuss how to calculate this change in preference because our focal point is our newly proposed method for transformation of 2×2 games. Again we considered the example of the Prisoners dilemma for illustration purposes.

	Corporate	*Defect*
Corporate	*3,3*	*1,4*
Defect	*4,1*	*2,2*

Figure 3:- Cardinal prisoners' dilemma

Let's suppose that a deal is struck between prison authorities and player one that if player one corporates they will decrease his sentence by one month and correspondingly if he defects they will increase his sentence in prison by one month. So his preference towards corporate will increase by one and towards defect also decrease by one.

	Corporate	*Defect*
Corporate	*3+1,3*	*1,4*
Defect	*4-1,1*	*2,2*

Figure 3.1:- Under operation

	Corporate	*Defect*
Corporate	*4,4*	*2,2*
Defect	*3,1*	*4,3*

Figure 4:- Newly transformed cardinal Asym dilemma.

For simplicity purposes we have taken the example of one month change, it could be any time period ($A = \{a_1, \dots, a_n\}$) depending upon the degree or intensity of preference change. It should also be noted that for games involving cardinal payoffs depending upon the intensity of change in preference a game may or may not be transformed in to a new game when a change in preference takes place.

Limitations and implications:-

In this paper we paid much attention to our newly proposed approach for potential transformation of both cardinal and ordinal 2×2 Games. While little attention was paid towards exemplification as a matter of fact there are no less than 144 2×2 Games [1]. But it is a field future researchers can exploit and transform both ordinal and cardinal 2×2 Games using our newly proposed method.

The purpose of this paper was threefold: to expand the 2×2 games transformation problem by allowing both cardinal and ordinal transformations, to present appropriate method capable of transforming both cardinal and ordinal games and to open a window for the potential development of new taxonomies and topologies.

Up until now most of the taxonomies and topologies have been developed by taking in to consideration only ordinal payoffs while cardinal payoffs have mostly being ignored. Our work will open a new window for the potential development of new taxonomies and topologies involving both cardinal and ordinal payoffs. Our method will be useful to resolve such problems where both cardinal and ordinal transformations are required. Our proposed method is not only good enough for transformation of 2×2 games with ordinal payoffs but also for games with cardinal payoffs (for mixed strategy games cardinal payoffs must be considered [26]) as well as for those games where one player is cardinal and the other is ordinal.

Appendix 1:-

Condition 1 (Both players preference towards s_1 increases will result in decrease toward s_2)

$P(s_1) \rightarrow Q(s_2)$
$Then\ for\ Player\ 1$: $[\ u_1(s_1)+A]$ & $[\ u_1(s_2)-A]$
$At\ the\ same\ time\ for\ Player\ 2$:$[\ u_2(s_1)+A]$& $[\ u_2(s_2)-A]$

Condition 2(Both players preference towards s_1 decreases will result in an increase toward s_2 for both players)

$Q(s_2) \rightarrow P(s_1)$
$Then$:
$For\ Player\ 1$: $[\ u_1(s_1)-A]$ & $[\ u_1(s_2)+A]$
$At\ the\ same\ time\ for\ Player\ 2$: $[\ u_2(s_1)-A]$ & $[\ u_2(s_2)+A]$

Condition 3(Only Player 1 preference towards s_1 increases will result in a decrease towards s_2 for player 1):

$P(s_1) \rightarrow Q(s_2)$
For player 1:
$[u_1(s_1)+A]$& $[u_1(s_2)-A]$
There will be no change in player 2 preferences.

Condition 4(Only Player 1 preference towards s_1 decreases will result in an increase towards s_2 for player 1):

$Q(s_2) \rightarrow P(s_1)$
$Then$: $[u_1(s_1)-A]$ & $[u_1(s_2)+A]$

Condition 5(Only Player 2 preference towards s_1 increases will result in a decrease towards s_2 for player 2):

$P(s_1) \rightarrow Q(s_2)$
$Then$: $[u_2(s_1)+A]$ & $[u_2(s_2)-A]$

Condition 6(Only Player 2 preference towards s_1 decreases will result in an increase towards s_2 for player 2):

$Q(s_2) \rightarrow P(s_1)$
$Then$: $[u_2(s_1)-A]$ & $[u_2(s_2)+A]$

Condition 7 (Player 1 preference towards s_1 increases and Player 2 preference towards s_1 decreases this will result in a decrease towards s_2 for player 1 and an increase towards s_2 for player 2)

For player 1: $P(s_1) \rightarrow Q(s_2) \wedge$ for player 2 $Q(s_2) \rightarrow P(s_1)$
$Then\ for\ Player\ 1$: $[\ u_1(s_1)+A]$ & $[\ u_1(s_2)-A]$
$At\ the\ same\ time\ for\ Player\ 2$: $[\ u_2(s_1)-A]$ & $[\ u_2(s_2)+A]$

Condition 8 (Player 1 preference towards s_1 decreases and Player 2 preference towards s_1 increases this will result in an increase towards s_2 for player 1 and in a decrease towards s_2 for player 2)

For player 2: $P(s_1) \rightarrow Q(s_2) \wedge$ for player 1 $Q(s_2) \rightarrow P(s_1)$
$For\ Player\ 1$: $[\ u_1(s_1)-A]$ & $[\ u_1(s_2)+A]$
$At\ the\ same\ time\ for\ Player\ 2$:$[\ u_2(s_1)+A]$& $[\ u_2(s_2)-A]$

Appendix 2:-

Periodic table for all 144 2×2 Games.

This table has been sourced from [25].

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