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RESEARCH ARTICLE

GAMMA ACTS.

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Abstract

Let S be a Γ-semigroup . We introduce the notion of gamma act over Γ-semigroup S and study some important properties of such acts, with this respect , we study gamma subacts , congruences and homomorphisms, of gamma acts further ,we give related basic results of gamma acts .And we will show the class of gamma acts is a generalization of S-acts and Γ-semigroups.

1. Introduction

The concept of S-act has been introduced as follows: if S is a semigroup, a nonempty set M is called a left S-act if there is a mapping from S×M into M and the following condition is satisfied : $s_1(s_2m) = (s_1s_2)m$ for all $s_1,s_2 \in S$ and $m \in M$ [1]. Every semigroup can be consider as act over itself. By a similar way we define right S-act .The S-act theory is a generalization of R-module theory .

The concept of Γ-semigroup has been introduced by M.K. Sen in 1981 as follows: if S and Γ are nonempty sets, S is called a Γ-semigroup if there is a mapping from S×Γ×S into S and the following condition is satisfied :$(s_1\alpha s_2)\beta s_3 = s_1\alpha (s_2\beta s_3)$ for all $s_1,s_2,s_3 \in S$ and $\alpha, \beta \in \Gamma$ [2]. A nonempty subset A of a Γ-semigroup S is called right ideal of S if $A\Gamma S \subseteq A$ where $A\Gamma S = \{ a\alpha s \mid a \in A , \alpha \in \Gamma \text{ and } s \in S \}$. And it is called a left ideal of S if $S\Gamma A \subseteq A$, A is called ideal of S if it is both a left and a right ideal of S.

Let S and T be Γ-semigroups under the same Γ **.** A mapping f : S → T is called a Γ –homomorphism if $f(s_1\alpha s_2) = f(s_1)\alpha f(s_2)$ for all $s_1,s_2 \in S$ and $\alpha \in \Gamma$.

2. Gamma acts

In this section we introduce and study the concepts of gamma act over a fixed Γ-semigroup .

Definition 2.1 Let S be a Γ-semigroup, A nonempty set M is called left S_Γ-act (denoted by $S_\Gamma M$) if there is a mapping from S×Γ×M into M written (s_1,α,s_2) by $s_1\alpha s_2$ such that the following condition is satisfied

$$(s_1\alpha s_2)\beta m = s_1\alpha(s_2\beta m) \quad \text{for all } s_1 ,s_2 \in S , \alpha ,\beta \in \Gamma \text{ and } m \in M$$

Similarity one can define a right gamma acts .

Definition 2.2 A left S_Γ-act M is called unitary if there exist $1 \in S$, $\alpha_\circ \in \Gamma$ such that $1\alpha_\circ m = m$ for all $m \in M$. We denote the element $1\alpha_\circ$ by $1_{\alpha_\circ}$ i.e $1_{\alpha_\circ} m = m$ for all $m \in M$.

Definition 2.3 Let M be left S_Γ-act . An element $\Theta \in M$ is called a zero of M if $s\alpha\Theta = \Theta$ for all $s \in S$ and $\alpha \in \Gamma$.

Note. An S_Γ-act M can have more than one zero elements, see example 2.6 (A) . And Every S_Γ-act M can be extended to an S_Γ-act with zero by taking the disjoint union $M \uplus \{ \Theta \}$,where $\{ \Theta \}$ is a one-element S_Γ-act with $s\alpha\Theta = \Theta$ for all $s \in S$ and $\alpha \in \Gamma$.

Definition 2.4 Let S and T be Γ-semigroups. A nonempty set M is called gamma biact denoted by $(T\text{-}S)_\Gamma$-biact if
1. M is a left T_Γ-act
2. M is a right S_Γ-act
3. $t\alpha(m\beta s) = (t\alpha m)\beta s$ for all $t \in T$, $\alpha ,\beta \in \Gamma$, $m \in M$ and $s \in S$.

Definition 2.5 Let M be $(T\text{-}S)_\Gamma$-biact. Then M is called unitary $(T\text{-}S)_\Gamma$-biact if
1. M is unitary left T_Γ-act , i.e there exist $1_T \in T$ and $\alpha_{0T} \in \Gamma$ such that $m = 1_T\alpha_{0T}m$ for all $m \in M$
2. M is unitary right S_Γ-act , , i.e there exist $1_S \in S$ and $\alpha_{0S} \in \Gamma$ such that $m = m\alpha_{0S}1_S$ for all $m \in M$

Note . All S_Γ-act in following , consider unitary left S_Γ-act unless otherwise we stated

In the following by many examples we illustrate the notion of gamma acts and show that the class of gamma acts is very wide.

Examples 2.6
A. Let $S = \mathbb{Z}$, $\Gamma = \mathbb{N}$, Then S is Γ-semigroup $(z_1,n,z_2) \to z_1.n.z_2$ with usual multiplication of integer numbers. Let M be a nonempty set .Then M is an S_Γ-act under the mapping from $S\times\Gamma\times M$ into M which define by $(z,n,m) \to m$ for all $s \in S$, $\alpha \in \Gamma$, $m \in M$.

B. If S is a Γ-semigroup and M a nonempty set. Any fixed element $m_\circ$ in M gives rise on S_Γ-act structure of M by the mapping $S\times\Gamma\times M \to M$ define by $(s,\alpha,m) \to m_\circ$ for all $s \in S$, $\alpha \in \Gamma$, $m \in M$

example (**B**) shows that any nonempty set can be concider as S_Γ-act for any Γ-semigroup S . In particular every singleton set is a one-element S_Γ-act.

C. Let S = { 5n+4 | n is a positive integer } , Γ = { 5n+1 | n is a positive integer } .Then S is a Γ-semigroup where $s_1\alpha s_2 = s_1+ \alpha+ s_2$. Now let M = { 5n | n is a positive integer } . Then M is an S_Γ-act where $s\alpha m = s + \alpha + m$, where + is the usual addition .

D. Let S be a Γ-semigroup. A polynomial in one indeterminate X with coefficients in S respect to Γ is to be an expression $P(X) = s\beta X$, $s \in S$, $\beta \in \Gamma$, where X is a any symbol . The set S[X] of all polynomials is a nonempty set becomes to an S_Γ-act under the mapping from $S\times\Gamma\times S[X]$ into S[X] define by $(r,\alpha,P[X]) \to (r\alpha s)\beta X$

E. Every Γ-semigroup S is an S_Γ-act with $s_1.\alpha.s_2$ being the Γ-semigroup structure in S .

F. Let S = [0 , 1] , $\Gamma = \{ \frac{1}{n} \mid$ n is a positive integer } and M = S , Then M is an S_Γ-act under usual multiplication of real numbers.

G. Consider the following two sets $S = \left\{ \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} \mid a , b \in \mathbb{R} \right\}$ and $\Gamma = \left\{ \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \mid x \in \mathbb{R} \right\}$ and let M = S .Then M is an S_Γ-act under the usual product of the matrices .

H. Let $S = \{ \varnothing , \{a\} ,\{b\} ,\{c\} ,\{a ,b\} ,\{b ,c\} ,\{a ,c\} ,\{a ,b ,c\}\}$, $\Gamma=\{\varnothing , \{a\} ,\{a ,b ,c\} \}$ and M = S .Then M is an S_Γ-act where $ABC = A\cap B\cap C$. for all $A ,C \in S$ and $C \in \Gamma$.

I. Let R be an Γ-ring and M is an R_Γ-module [3]. It is clear that any R_Γ-module is R_Γ-act .

J. Let M be an R-module , define a mapping . : R×R×M → M , By (r,s,m) → (rs)m being the R-module structure of M , Then M is an R_R-act.

K. Let S be a Γ-semigroup and I be a left ideal of S . Then I is a left S_Γ-act under the mapping . :S×Γ×I → I define by (s,α,r) → s.α.r , for all s ∈ S , α ∈ Γ and r ∈ I

L. Let M be an arbitrary semigroup and S an arbitrary nonempty subset of ℤ, Then M is $\mathbb{Z}_S$-act under the mapping ℤ×S×M → M define by (n,n',m) → (n.n')m .

M. let ℝ be set of of all real numbers. $\mathbb{R}^n$ is an $\mathbb{R}_\mathbb{R}$-act . by the mapping from ℝ× ℝ× $\mathbb{R}^n$ into $\mathbb{R}^n$ define $(r_1,r_2,(x_i)_{i=1}^{n}) \to (r_1 r_2 x_i)_{i=1}^{n}$ for all $r_1 ,r_2 \in \mathbb{R}$ and $(x_i)_{i=1}^{n} \in \mathbb{R}^n$.

N. Let S be Γ-semigroup. Then S×ℤ = {(s,z) | s ∈ S , z ∈ ℤ} is an S_Γ-act by the mapping from S×Γ×(S×ℤ) into S×ℤ define by (s,α,(s',z)) → (sαs',z) .

O. Let G be a group , Λ_1 , Λ_2 two index sets and Γ the collection of some $\Lambda_1 \times\Lambda_2$ matrices over G° = G∪{0} ,the group with zero .Let S = Γ .Then it is easy to see S is a Γ-semgroup . M = { $(a)_{ij}$ | i ∈ Λ_1 ,j ∈ Λ_2 and (a_{ij}) the $\Lambda_1 \times\Lambda_2$ matrix over G° with a_{ij} = a and 0 otherwise } .For any $(a)_{ij}$,$(b)_{jk}$,$(c)_{kv}$ ∈ M and α = (p_{ji}) , β = (q_{ji}) ∈ Γ we define $(a)_{ij}\alpha(b)_{jk} = (ap_{jk}b)_{ik}$. Then it is easy verified that $[(a)_{ij}\alpha(b)_{jk}]\beta(c)_{kv} = (a)_{ij}\alpha[(b)_{jk}\beta(c)_{kv}]$ .Thus M is S_Γ-act .

P. Let S = { a, b, c, d, e} , Γ={α , β } and M = S×S . Put the binary operations in the tables below

α	a	b	c	d	e
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c
e	e	a	b	c	d

β	a	b	c	d	e
a	b	c	d	e	a
b	c	d	e	a	b
c	d	e	a	b	c
d	e	a	b	c	d
e	a	b	c	d	e

And consider the mapping from S×Γ×M into M by $(s,\alpha,(s_1,s_2) \to (s\alpha s_1,s\alpha s_2)$. Since $(s_1\alpha s_2)\beta m = s_1\alpha(s_2\beta m)$ for all $s_1 ,s_2 \in S$, α ,β ∈ Γ and m ∈ M . M is an S_Γ-act .

The following proposition gives a new example of old ones .

Proposition 2.7 Let M be an S_Γ-act , and P(M) the power set of M ,Then P(M) is an S_Γ-act .

proof **:** Consider the mapping S×Γ×P(M) → P(M) define by (s,α,X) = sαX where sαX = { sαx | x ∈ X , s ∈ S ,α ∈ Γ}. Then for all $s_1 ,s_2 \in S$, α ,β ∈ Γ and X ∈ P(M) we have $s_1\alpha(s_2\beta X) = s_1\alpha\{ s_2\beta x \mid x \in X\ s \in S ,\alpha \in \Gamma \}$ = $\{ s_1\alpha(s_2\beta x) \mid x \in X ,s \in S ,\alpha \in \Gamma \} = \{ (s_1\alpha s_2)\beta x) \mid x \in X, s \in S ,\alpha \in \Gamma \} = (s_1\alpha s_2)\beta X$ □

Example 2.8 It is well-known that both ℤ and ℚ is an ℕ-semigroups under the usual multiplication .Then ℝ is an $(\mathbb{Z}-\mathbb{Q})\mathbb{N}-\text{biact}$. As well as $(\mathbb{Q}-\mathbb{Z})\mathbb{N}-\text{biact}$.

The following example shows that if M is an S-act, then M is an S_Γ-act for every nonempty set Γ .

Examples 2.9 Let M be right S-act and Γ be any nonempty set. Define S×Γ×S → S by (s,α,s') ↦ ss' for all s ,s' ∈ S and α ∈ Γ .Then
1. S is Γ-semgroup indeed $(s_1\alpha s_2)\beta s_3 = s_1\alpha(s_2\beta s_3)$ for all $s_1 ,s_2 ,s_3 \in S$ and α ,β ∈ Γ .
2. M is S_Γ-act, in fact (mαs)βs' = mα(sβs') for all s ,s' ∈ S , α ,β ∈ Γ and m ∈ M .

In the following example. We show that the converse of Examples 2.9 may not be true in general and hence S_Γ-acts are a proper generalization of S-acts.

Examples 2.10

1. Let M be a set of all negative rational numbers .It is clear that M is not M-act under usual product of rational numbers . Let $\Gamma = \{ -\frac{1}{p} \mid p \text{ is prime} \}$ .Let a ,b ,c ∈ M and α ,β ∈ Γ .Now if aαb is equal to the usual product of rational numbers then aαb ∈ M and (aαb)βc = aα(bβc) .hence M is M_Γ-act .

2. Let M = { i ,0 ,-i } and S = Γ = M . Then M is S_Γ-act under the multiplication over complex numbers while M is not S-act under the multiplication .

Note. The class of gamma acts is very wide .In the following example we see that a nonempty set to be a gamma act depends on the mapping of multiplication. as in the following example.

Example 2.11 Let M be the set of all odd integer numbers . then M is not $\mathbb{Z}_\mathbb{Z}$-act under the usual multiplication of integer numbers .

3. Gamma subacts and congruences

In this section we study gamma subacts, congruences of gamma act and investigate their properties .

Definition 3.1 Let M be an S_Γ-act , A nonempty subset N of M is called S_Γ-subact if SΓN ⊆ N , Where SΓN = { sαn | s ∈ S , α ∈ Γ, n ∈ N } . In this case we write N ≤ M .

Note . clearly M be a trivial gamma subact on M and if M with zero element Θ ,then { Θ } be also trivial gamma subact ,and any left ideal of Γ-semgroup S is an S_Γ-subact on S .

Example 3.2 Clearly that ℤ is $\mathbb{Z}_\mathbb{N}$-act , and $\mathbb{Z}_e$ is the set of all even integers. Then $\mathbb{Z}_e$ is an $\mathbb{Z}_\mathbb{N}$-subact .

Example 3.3 Let S = ℤ , Γ = ℕ , Then S is a Γ-semigroup by the mapping $(z_1,n,z_2) \to z_1.n.z_2$ be usual multiplication . Let M = { 1, 2, 3, 4, 5, 6 }, Then M is S_Γ-act by the mapping S×Γ×M → M define by (z,n,m) → 2 . Then any nonempty subset of M that contains 2 is S_Γ-subact of M .

Example 3.4 consider the gamma act in Examples (2.6) (m) , where n = 2 .Then if N = { (x,0) | x ∈ ℝ} and M = { (0,y) | y ∈ ℝ }, then M and N are S_Γ-subact .

Proposition 3.5 Let M be S_Γ-act , let { N_i | i ∈ I } be collection of S_Γ-subact in M . Then.
1. If $\bigcap_{i\in I} N_i$ is a nonempty, then $\bigcap_{i\in I} N_i \leq M$.
2. $\bigcup_{i\in I} N_i \leq M$.

Proof 1. Since $N_i \leq M$ for all i ∈ I ,then $N_i \neq \varnothing$ for all i ∈ I , then we get $\bigcap_{i\in I} N_i \neq \varnothing$ .Now Let $x \in \bigcap_{i\in I} N_i$, s ∈ S and α ∈ Γ . We get $x \in N_i$ for all i ∈ I .Since $s\alpha x \in N_i$ for all i ∈ I . Then $s\alpha x \in \bigcap_{i\in I} N_i$. we get $\bigcap_{i\in I} N_i \leq M$
2. $N_i \leq M$ for all i ∈ I ,then $N_i \neq \varnothing$ for all i ∈ I , then we get $\bigcup_{i\in I} N_i \neq \varnothing$. Now let $x \in \bigcup_{i\in I} N_i$, s ∈ S and α ∈ Γ . Then there is some i° ∈ I such that x ∈ Ni° , implies $s\alpha x \in \bigcup_{i\in I} N_i$.Then we get $\bigcup_{i\in I} N_i \leq M$ □

Proposition 3.6 Let M be an S_Γ-act , X is a nonempty subset from M. Then the set define by $[X]_M := \bigcap\{B \mid X \subseteq B , B \leq M\}$, is the smallest S_Γ-subact of M contains X.

Proof **.** This clearly by proposition(3.5), that $[X]_M \leq M$. Now let N ≤ M and X ⊆ N , by definition of $[X]_M$ we get $[X]_M \subseteq N$ .Then $[X]_M$ is smallest S_Γ-subact of M which contains X. □

Let M be an S_Γ-act , N ≤ M . Define N : M = { s ∈ S | sαm ∈ N for all α ∈ Γ and m ∈ M } .

If N : M is a nonempty subset of S , then it is easy to see that N : M is a left ideal of a Γ-semgroup S .

The proof of the following proposition follows from the definition .

Proposition 3.7 Let M be an S_Γ-act , N and L be two S_Γ-subacts and A , B are two nonempty subsets of M . Then

1. if $A \subseteq B$ implies that $(N : B) \subseteq (N : A)$.
2. $(N \cap L : A) = (N : A) \cap (N : A)$.

Definition 3.8 Let M be an S_Γ-act**.** An equivalence relation ρ on M is called a congruence on M, if $(m_1,m_2) \in \rho$, implies that $(s\alpha m_1, s\alpha m_2) \in \rho$ for all $s \in S$, $\alpha \in \Gamma$ and $m_1, m_2 \in M$**.**

Example 3.9 Let M be an S_Γ-act , Then $I_M = \{ (m,m) \mid m \in M \}$ is a trivial congruence , M×M is a universal congruence on M .

Example 3.10 consider the gamma act in Examples(2.6)(a) **,** with M={ a , b , c } , put ρ = { (a ,a) , (b ,b) , (c ,c), (a ,b) , (b ,a) } ⊆ M×M , then ρ is an congruence on M .

Proposition 3.11 Let M be S_Γ-act , ρ be congruence on M . Then the set denoted by M/ρ of xρ where xρ the equivalent class contains x is an S_Γ-act .

Proof. Define the mapping $S\times\Gamma\times M/\rho \rightarrow M/\rho$ by $(s,\alpha,m\rho) \rightarrow (s\alpha m)\rho$, let $s_1 , s_2 \in S$, $\alpha , \beta \in \Gamma$. First to show the mapping be well define , let $x\rho = x'\rho$ then $(x ,x') \in \rho$ and $(s\alpha x , s\alpha x') \in \rho$, implies $(s\alpha x)\rho = (s\alpha x')\rho$. and $(s_1\alpha s_2)\beta x\rho = ((s_1\alpha s_2)\beta x)\rho = (s_1\alpha(s_2\beta x))\rho = s_1\alpha(s_2\beta x)\rho = s_1\alpha(s_2\beta x\rho)$. Then M/ρ is S_Γ-act □

Note. M/ρ is called quotient gamma act under congruence ρ on M .

Proposition 3.12 Let M be S_Γ-act, $\{ \rho_\alpha \mid \alpha \in \Omega \}$ be a family of congruences on M, then $\cap\rho_\alpha$ is the largest congruence on M contained in ρ_α for all $\alpha \in \Omega$.

Proof Clearly that $\cap\rho_\alpha$ is equivalent relation on M . Now let $(x,y) \in \cap\rho_\alpha$, $s \in S$ and $\alpha \in \Gamma$, then $(x,y) \in \rho_\alpha$ for all $\alpha \in \Omega$ and $(s\alpha x, s\alpha y) \in \rho_\alpha$ for all $\alpha \in \Omega$, then $(s\alpha x, s\alpha y) \in \cap\rho_\alpha$. we get $\cap\rho_\alpha$ be congruence on M . Now let σ be a congruence on M contained in ρ_α for all $\alpha \in \Omega$. Let $(x,y) \in \sigma$ then $(x,y) \in \cap\rho_\alpha$, we get $\cap\rho_\alpha$ is largest congruence on M contained in ρ_α for all $\alpha \in \Omega$ □

Let M be S_Γ-act , and H be a nonempty subset of M . Define

$$\ell_S(H) = \{ (s,t) \in S\times S \mid s\alpha h = t\alpha h \text{ for all } \alpha \in \Gamma \text{ and } h \in H \}$$

it is clear that $\ell_S(H)$ is a congruence on S_Γ-act S and if M with zero element Θ ,then $\ell_S(\Theta) = S\times S$.

Proposition 3.13 Let M be an S_Γ-act , and A , B are two nonempty subsets of M . Then
1. if $A \subseteq B$ implies that $\ell_S(B) \subseteq (\ell_S(A)$.
2. $\ell_S(A\cup B) = \ell_S(A) \cup \ell_S(B)$.

Definitoin 3.14 Let M be $(S\text{-}T)_\Gamma$-biact. An equivalence relation ρ on M is called a congruence on M if $(m_1,m_2) \in \rho$, implies $(t\alpha m_1\alpha s, t\alpha m_2\alpha s) \in \rho$,for all $s \in S$, $\alpha \in \Gamma$, $m_1 , m_2 \in M$ and $t \in T$.

4. Homomorphisms gamma acts.

In this section we study the homomorphisms of gamma acts. In particular we investigate the behavior of gamma subacts and congruences under their homomorphisms.

Definition 4.1 Let M and N be two S_Γ-acts . A mapping $f : M\rightarrow N$ is called S_Γ-homomorphism if $f(s\alpha m) = s\alpha f(m)$. for all $s \in S$, $\alpha \in \Gamma$ and $m \in M$. A homomorphism $f : M\rightarrow N$ is called

1. monomorphism if f is injective mapping .
2. epimorphism if f is surjective mapping
3. isomorphism if f is bijective mapping .

If f : M → N is an S_Γ-isomorphism, then we called M isomorphic to N denoted by M ≅ N. The set of all S_Γ-homomorphisms from M into N denote by $Hom_{S_\Gamma}(M,N)$. If M = N, then $Hom_{S_\Gamma}(M,N)$ denote by $End_{S_\Gamma}(M)$.

Definition 4.2 Let f: M → N be S_Γ-Homomorphism . Then we define the kernel and the image of f as follows .
1. ker(f) = { (m_1,m_2) ∈ M×M | $f(m_1) = f(m_2)$ } .
2. Im(f) = { n ∈ N | there is m ∈ M such that f(m) = n }.

Definition 4.3 Let M and N be $(T\text{-}S)_\Gamma$-biact . A homomorphism f : M → N is called $(T\text{-}S)_\Gamma$-homomorphism , if
1. f : M → N be T_Γ-homomorphism
2. f : M → N be S_Γ-homomorphism
3. f(tαm)βs = tαf(mβs) , for all s ∈ S , α ,β ∈Γ , m ∈ M and t ∈ T .

Example 4.4 Let M be an S_Γ-act . and N ≤ M . Then the mapping i : N → M define by i(n) = n for all n ∈ N, is an S_Γ-monomorphism .

Proposition 4.5 Let f :M→N be S_Γ-Homomorphism. Then
1. Ker(f) is a congruence on M.
2. If G ≤ M , then f(G) = { f(m) | m ∈ G } ≤ N. In particular f (M) ≤ N .

Proof. 1. It is clear that Ker(f) is equivalent relation on M . Now let (m_1,m_2) ∈ Ker(f) , s ∈ S and α ∈ Γ . implies $f(s\alpha m_1) = s\alpha f(m_1) = s\alpha f(m_2) = f(s\alpha m_2)$, thus Ker(f) is a congruence on M
2. clearly f(G) is a nonempty subset of N . Now let s ∈ S , α ∈ Γ and n ∈ f(G) . Then there is m ∈ G such that f(m) = n , then sαn = sαf(m) = f(sαm) ∈ f(G). □

The following example shows the converse of proposition(4.5) 1. is also true .

Example 4.6 Let M be S_Γ-act , ρ be a congruence on M . Then the mapping π_ρ : M → M/ρ define by π_ρ(m) = mρ for all m ∈ M is called canonical map , and clearly that π_ρ is an S_Γ-epimorphism and ker(π_ρ) = ρ

Proposition 4.7 Let M be an S_Γ-act, and ρ a congruence on M . Then
1. if N ≤ M ,then $N/\rho_N \le M/\rho$.
2. if W≤ M/ρ , then there exist L ≤ M such that $W = L/\rho_L$, where $\rho_L = \rho \cap (L\times L)$

Proof. 1. It is clear .
2. clearly $\pi_\rho^{-1}(W) \le M$, and let $\pi_\rho^{-1}(W) = L$ .Then by1, we get $L/\rho_L \le M/\rho$ and it is clear to see that $L/\rho_L = W$. □

Proposition 4.8 Let f:M→N be S_Γ-homomorpohism , and ρ be congruence on M then ρ_f ={ (f(x),f(y)) | (x,y) ∈ ρ } is a congruence on f(M) .

Proof . It is straightforward to check that ρ_f is an equivalent relation on f(M). Now let (f(x),f(y)) ∈ ρ_f , s ∈ S , α ∈ Γ. Since (x,y) ∈ ρ we get (sαx,sαy) ∈ ρ , (f(sαx),f(sαy) = (sαf(x), sαf(y)) ∈ ρ_f. Then ρ_f is a congruence on (M) . □

Proposition 4.9 Let f:M→N be S_Γ-homomorphism , ρ a congrunce on M , and ρ* on f(M) with ρ_f ⊆ ρ* .Then there exsits a homomorphism from M/ρ to f(M)/ρ* and f(M)/ρ* is an epimorphic image of M/ρ.

Proof. Let f*:M/ρ→f(M)/ρ* define by xρ → f(x)ρ*. If xρ = yρ then (x,y) ∈ ρ and (f(x),f(y)) ∈ ρ_f ⊆ρ*, then (f(x),f(y)) ∈ ρ* and f(x)ρ* = f(y)ρ* , f* is well define . Let xρ∈M/ρ ,s∈S α∈Γ. Then f*(sα(x)ρ) = f*((sαx)ρ) = (sαx)ρ* = sα(x)ρ* = sαf*(xρ), Then f* is S_Γ-homomorphism act from M/ρ to f(M)/ρ* □

Lemma 4.10 Let S and R be Γ-semigroups and let Φ : R→S be Γ-homomorphism. If M is S_Γ-act, then M is R_Γ-act.

Proof . Define a mapping R×Γ×M → M by (r,α,m) → Φ(r)αm .Then $(r_1\alpha r_2)\circ\beta\circ m = \Phi(r_1\alpha r_2)\beta m = (\Phi(r_1)\alpha\, \Phi(r_2))\beta m = \Phi(r_1)\alpha\, (\Phi(r_2)\beta m) = \Phi(r_1)\alpha(r_2\circ\beta\circ m) = r_1\circ\alpha(r_2\circ\beta\circ m)$ for all r_1 , r_2 ∈ R , α ,β ∈ Γ , m ∈ M .

Proposition 4.11 Let M , N be S_Γ-acts . Then $Hom_{S_\Gamma}(M,N)$ is an S_Γ-act.

Proof. Consider the mapping $S\times\Gamma\times Hom_{S_\Gamma}(M,N)\rightarrow Hom_{S_\Gamma}(M,N)$ which define by $(s,\alpha,\Phi) = s\alpha\Phi$, where $s\alpha\Phi(m) = s\alpha\Phi(m)$, $m \in M$. Since M be S_Γ-act, clearly that $Hom_{S_\Gamma}(M,N)$is S_Γ-act □

Proposition 4.12 Let $f : M \rightarrow N$ be an S_Γ-homomorphism, and ρ a congruence on M such that $\rho \subseteq Ker(f)$. Then there exists unique S_Γ-homomorphism$\bar{f} : M/\rho \rightarrow N$, $\bar{f}\pi\rho = f$ and $\bar{f}$ is S_Γ-epimorphism if and only if f $S_\Gamma$epimorphism .
Proof. First we must show $\bar{f}$ is well define . Let $x\rho, y\rho \in M/\rho$ such that $x\rho = y\rho$, then $(x,y) \in \rho \subseteq Ker(f)$ then $f(x) = f(y)$. Then $\bar{f}(x\rho) = \bar{f}(y\rho)$, this shows that $\bar{f}$ is well defined. $\bar{f}(s\alpha x\rho) = \bar{f}((s\alpha x)\rho) = f(s\alpha x) = s\alpha f(x) = s\alpha\bar{f}(x\rho)$, then $\bar{f}$ is S_Γ-homomorphism and clearly $Im(f) = Im(\bar{f})$. Let $x \in M$, the $(\bar{f}\pi\rho)(x) = \bar{f}(\pi\rho(x)) = \bar{f}(x\rho) = f(x)$
for all $x \in M$ then $\bar{f}\pi\rho = f$ and clearly $\bar{f}$ is a uniqe and $\bar{f}$ S_Γ-epimorphism if and only if f S_Γ-epimorphism . □

From proposition(4.12) , if $f : M \rightarrow N$ is an S_Γ-homomorphism , then $M/ker(f) \cong Im(f)$

Theorem 4.13 Let ρ_1 and ρ_2 be two congruences on S_Γ-act M. with $\rho_1 \subseteq \rho_2$. Define

$$\rho_2/\rho_1 = \{ (x\rho_1,y\rho_1) \in (M/\rho_1)\times(M/\rho_1) \mid (x,y) \in \rho_2\}. \text{ Then}$$

1. ρ_2/ρ_1 is a congruence on M/ρ_1.
2. $(M/\rho_1)/(\rho_2/\rho_1) \cong M/\rho_2$.

Proof. 1. First we show that ρ_2/ρ_1 is an equivalent relation on M/ρ_1 , let $x\rho_1\in M/\rho_1$, implies $(x\rho_1,x\rho_1)\in \rho_2/\rho_1$. If $(x\rho1,y\rho1)\in M/\rho_1$ then $(x,y) \in \rho_2$ and $(y,x) \in \rho_2$ therefore $(y\rho_1,x\rho_1) \in \rho_2/\rho_1$. Next if $(x\rho_1,y\rho_1)$ and $(y\rho_1,z\rho_1)$ then $(x,y),(y,z) \in \rho_2$ we get $(x,z) \in \rho_2$,then $(x\rho_1,z\rho_1) \in \rho_2/\rho_1$. Finally let $s \in S$, $\alpha \in \Gamma$, $(x\rho_1,y\rho_1) \in \rho_2/\rho_1$, then $(x,y) \in\rho_2$ and $(s\alpha x,s\alpha y) \in\rho_2$ implies that $(s\alpha x\rho_1,s\alpha y\rho_1) \in \rho_2/\rho_1$.
2 . Define Φ: $(M/\rho_1)/(\rho_2/\rho_1)\rightarrow M/\rho_2$ by $\Phi((x\rho_1)(\rho_2/\rho_1)) = x\rho_2$ for all $x \in M$. If $(x\rho_1)(\rho_2/\rho_1) = (y\rho_1)(\rho_2/\rho_1)$ then $(x\rho_1,y\rho_1) \in \rho_2/\rho_1$ and $(x,y) \in \rho_2$ implies $x\rho_2 = y\rho_2$. let $x\rho_2\in M/\rho_2$ then $x \in M$, $x\rho_1\in M/\rho_1$, $(x\rho_1)(\rho_2/\rho_1) \in (M/\rho_1)/(\rho_2/\rho_1)$ such that $\Phi((x\rho_1)(\rho_2/\rho_1)) = x\rho_2$ and hence Φ is onto ,let $\Phi((x\rho_1)(\rho_2/\rho_1)) = \Phi((y\rho_1)(\rho_2/\rho_1))$, implies that $x\rho_2 = y\rho_2$, $(x,y) \in \rho_2$ then $(x\rho_1,y\rho_1) \in \rho_2/\rho_1$ and $(x\rho_1)(\rho_2/\rho_1) = (y\rho_1)(\rho_2/\rho_1)$ this shows Φ is injective. Finally, let $s \in S$, $\alpha \in \Gamma$ and $(x\rho)(\rho_2/\rho_1) \in (M/\rho_1)/(\rho_2/\rho_1)$, then $\Phi(s\alpha(x\rho_1)(\rho_2/\rho_1)) = \Phi((s\alpha x)\rho_1)(\rho_2/\rho_1)) = (s\alpha x)\rho_2 = s\alpha(x)\rho_2 = s\alpha\Phi((x\rho_1)\rho_2/\rho_1))$.Then $(M/\rho_1)/(\rho_2/\rho_1) \cong M/\rho_2$ □

5. Finite generated, cyclic and simple gamma acts.

Let M be an S_Γ-act and X a nonempty subset of M we have proved that $[X]_M$ is the smallest S_Γ-subact of M which contains X .Note that $[X]_M \coloneqq \bigcap\{B \mid X \subseteq B , B \leq M \}$ is the S_Γ-subact generated by X . In the following proposition we describe $[X]_M$ in terms of their elements .

Proposition 5.1 Let M be an an S_Γ-act and X a nonempty subset of M . Then $[X]_M = \bigcup_{u\in X} S\Gamma u$ where
$S\Gamma u = \{ s\alpha u \mid s \in S \text{ and } \alpha \in \Gamma \}$.

Proof. Let $W = \bigcup_{u\in X} S\Gamma u.$, $x \in W$, $s \in S$ and $\alpha \in \Gamma$. There is $s' \in S$, $\alpha' \in \Gamma$, $u_\circ \in X$ such that $x = s'\alpha'u_\circ$, then $s\alpha x = s\alpha(s'\alpha'u_\circ) = (s\alpha s')\alpha'u_\circ \in S\Gamma u_\circ \subseteq W$, we have $W \leq M$. Since for all $x \in X$, $x = 1\alpha_\circ x \in S\Gamma x \subseteq W$ then $X \subseteq W$. By definition of $[X]_M$ we get $[X]_M \subseteq W$. Now if $x \in W$, then $x = s\alpha u' \in [X]_M$ where $s \in S$, $\alpha \in \Gamma$ and $u' \in X$, then $W \subseteq [X]_M$. we get $W = [X]_M$.

Proposition 5.2 Let M be an S_Γ-act and A, B nonempty subsets of M .Then
1. $[A\cap B] \subseteq [A]\cap[B]$
2. $[A\cup B] = [A]\cup[B]$
3. if $f : M \rightarrow N$ is an S_Γ-homomorphism , then $f([A]_M) = [f(A)]_N$
4. if $f : M \rightarrow N$ is an S_Γ-epimorphism and $\varnothing\neq C \subseteq N$, then $[\mathbf{f^{-1}}(C)]_M \subseteq \mathbf{f^{-1}}([C]_N)$

Proof. 1. Let $x \in [A\cap B]$ then there is $s \in S$,$\alpha \in \Gamma$, $x' \in A\cap B$ such that $x = s\alpha x'$, then $x = s\alpha x' \in [A]$, $x = s\alpha x' \in [B]$. Then $x = s\alpha x' \in [A]\cap[B]$, we get $[A\cap B]\subseteq[A]\cap[B]$.

2. $[A\cup B]=\cup_{u\in A\cup B} S\Gamma u=(\cup_{u\in A} S\Gamma u)\cup(\cup_{u\in B} S\Gamma u)=[A]\cup[B]$ .We get $[A\cup B]=[A]\cup[B]$.
3. Clearly that (A) is nonempty and since $[A]_M \leq M$, then $f([A]_M) \leq N$ by proposition 4.5 and $f([X]_M) = \{ f(s\alpha x) \mid s\alpha x \in [x]_M \}$, $[f(X)]_N = \{ s\alpha f(x) \mid x \in X \}$ and since f is S_Γ-homomorphism we get $\mathbf{f([A]_M)=[f(A)]_N}$
4. it is clear $\mathbf{f^{-1}}$(C) is a nonempty subset of N, $[f^{-1}(C)]_M = \{ s\alpha x \mid s \in S , \alpha \in \Gamma \text{ and } x \in f^{-1}(C)\}$, $f^{-1}([C]_N)= \{ x \in M \mid f(x) \in [C]_N \}$. If $x \in [f^{-1}(C)]_M$ then there exist $s \in S$, $\alpha \in \Gamma$ and $x'\in f^{-1}(C)$ such that $x = s\alpha x'$, then $f(x) = f(s\alpha x') = s\alpha f(x') \in [C]_N$. □

Definitoin 5.3 A nonempty subset U of S_Γ-act M is said to a set of generating elements or generating set of M if M = [U] . We say that M is finitely generated if [U] = M for some subset U of M which $|U| < \infty$. And M is a cyclic if M = [{u}] for some u ∈ M . In particular M = [M] .

Proposition 5.4 Let f:M→N be S_Γ-homomorphism .Then
1. If M is finitely generating (cyclic), then (M) is finitely generating (cyclic) .
2. If M = [U] and **Φ:**M→N be S_Γ-homomorphism , then if f(u) = Φ(u) for all u ∈ U implies f = Φ .

Proof. 1. Follows from Proposition (5.2).
2 . For m ∈ M , $f(m) = f(s\alpha h) = s\alpha f(h) = s\alpha\Phi(h) = (s\alpha h) = \Phi(m)$ for some s ∈ S, α ∈ Γ , h ∈ U . □

Note . Let M be an S_Γ-act and let m°∈M . Then <m°> = SΓm° = { sαm° | s ∈ S , α ∈ Γ } is a cyclic S_Γ-subact of M generated by m° .

Example 5.5 Consider the gamma act in example(2.6)(m) where n = 2 .Then [(1,0)]={ (sα,0) | s ∈ S , α ∈ Γ } and [(0,1)] = { (0,sα) | s ∈ S , α ∈ Γ } .

Example 5.6 Consider the gamma act in example(2.4) (H) .Then [{a}]={ AB{a} | A ∈ S , B ∈ Γ } = {∅ , {a} } and [{a,b,c}] = { AB{a,b,c} | A ∈ S , B ∈ Γ } = S . We get S_Γ-act S be cyclic S_Γ-act generated by {a ,b ,c} .

Example 5.7 It is well-known that $\mathbb{Z}$ is $\mathbb{Z}_\mathbb{N}$-act . A $\mathbb{Z}_\mathbb{N}$-subact $\mathbb{Z}_e$ is cyclic.
.
Definition 5.8 Let M be an S_Γ-act . Then
1. M is a simple S_Γ-act , if it contain no gamma subact other than M .
2. M is a 1-simple S_Γ-act , if it contain no gamma subact other than M and 1-element gamma subacts .

Examples 5.9 clearly one element S_Γ-act is simple

Clearly every simple is a 1-simple ,but the converse may not true as in the following example .

Let $S = \mathbb{Q}$, $\Gamma = \mathbb{Z}$, $M = \mathbb{Q}$. Then M is a Γ -semigroup by usual multiplication of numbers . i.e $(r_1,z,r_2) \to r_1.z.r_2$ be usual multiplication of numbers . Let N = {0} is clear that N ≤ M but N ≠ M then M is not simple S_Γ-act but 1-simple since . let N be non singleton S_Γ-subact of M. Let 0 ≠ x ∈ N, then $\frac{1}{x} \in S$ and $\frac{1}{x}.1.x = 1 \in N$. Now to show N = M let y ∈ M = S then y.1.1 = y ∈ N .we get N = M.

Proposition 5.10 Let M be an S_Γ-act.If M is simple. Then M = SΓm for every m ∈ M .

Now we give condition under which cyclic gamma acts are simple .

Proposition 5.11 Let M be cyclic S_Γ-act generated by u , and ρ be a congruence on M . Then the cyclic gamma act M/ρ is simple if and only if $u\rho \cap S\Gamma m \neq \emptyset$ for any m ∈ M .

Proof. Assume that M/ρ be simple, $\pi_\rho : M \to M/\rho$ be the canonical epimorphism and m ∈ M . Then $\pi_\rho(S\Gamma m)$ is a gamma subact of M/ρ .Since M/ρ is a simple we have $\pi_\rho(S\Gamma m) = M/\rho$ .Hence there exist x ∈ SΓm such that $\pi_\rho(x) = u\rho$ .Thus $x \in u\rho$ and $u\rho \cap S\Gamma m \neq \emptyset$.
Conversely. Let N ≤ M/ρ and $t \in \pi_\rho^{-1}(N)$ .By hypothesis there exist s ∈ S , α ∈ Γ such that sαt ∈ uρ . Now $u\rho = \pi_\rho(s\alpha t) = s\alpha\pi_\rho(t) \in N$. This implies M/ρ = SΓuρ ≤ N. Hence N = M/ρ . □

Definition 5.12 Let M be an S_Γ-act and $N \leq M$ .A Rees congruence ρ_N is a congruence on M define by a ρ_Nb if and only if a , b in N or a = b . We denote the resulting factor by M/N and call it the Rees factor of M by the gamma subact N. Clearly M/N has a zero element which is the class consisting of N, all other class are one-element sets

The following statement is a corollary of the previous proposition. And can also be obtained straightforward from definition of a simple act.

Proposition 5.13 Let N be a gamma subact of M . The Rees factor M/N is a simple if and only if N = M

Definition 5.14 Let M be S_Γ-act. M is called decomposable, if there are two sub act B, C of M such that $M = A \cup B$, $A \cap B = \varnothing$, otherwise M is call indecomposable.

Proposition 5.15 Every cyclic S_Γ-act is indecomposable .

Proof. let M = [{u}] and let $M = A \cup B$ ($A, B \leq M$) $u \in M = A \cup \underline{B}$ then either $a \in A$ or $a \in B$. If $a \in A$ then $M = [\{a\}] \subseteq A$. Then M = A, or $a \in B$ implies M = B ◻

Lemma 5.16 Let M be S_Γ-act, I be nonempty set, A_i be indecomposable S_Γ-subact of M for all $i \in I$, and $\bigcap_{i\in I} Ai \neq \varnothing$. Then$\bigcup_{i\in I} Ai$ is indecomposable S_Γ-subact .

Proof. Clearly $\bigcup_{i\in I} Ai$ is an S_Γ-subact from M . Assume there exists A decomposition $\bigcup_{i\in I} Ai = B \cup C$, let $x \in \bigcap_{i\in I} Ai \subseteq \bigcup_{i\in I} Ai = B \cup C$ with $B \cap C = \varnothing$. Then either $x \in \bigcup_{i\in I} Ai \cap B$ or $x \in \bigcup_{i\in I} Ai \cap C$. And since $A_i = A_i \cap (B \cup C) = (A_i \cup B) \cap (A_i \cup C) = \varnothing$, a **contradiction**! ◻

Definition 5.17 Let M be $(T\text{-}S)_\Gamma$-biact . A nonempty subset N of M is called
1. A right $(T\text{-}S)_\Gamma$-subbiact , if N is S_Γ-subact on M , i.e. $N\Gamma S \subseteq N$.
2. A left $(T\text{-}S)_\Gamma$-subbiact , if N is T_Γ-subact on M , i.e $T\Gamma N \subseteq N$.
3. $(T\text{-}S)_\Gamma$-subbiact, if it is right and left $(T\text{-}S)_\Gamma$-subbiact ,i.e $T\Gamma N \subseteq N$ and $N\Gamma N \subseteq N$

Definition 5.18 . Let M be $(T\text{-}S)_\Gamma$-biact . Then M is called
1. Right simple $(T\text{-}S)_\Gamma$-biact . If it has not right $(T\text{-}S)_\Gamma$-subbiact other than M
2. Left simple $(T\text{-}S)_\Gamma$-biact . If it has not left $(T\text{-}S)_\Gamma$-subbiact other than M
3. Simple $(T\text{-}S)_\Gamma$-biact . If it has not neither left nor right $(T\text{-}S)_\Gamma$-subbiact other than M

Theorem 5.19 Let M be $(T\text{-}S)_\Gamma$-biact . If M is a left or right simple $(T\text{-}S)_\Gamma$-biact , then M is a simple $(T\text{-}S)_\Gamma$-biact .

Proof. consider M is right simple $(T\text{-}S)_\Gamma$-biact , Let N be $(T\text{-}S)_\Gamma$-subbiact on M .Then N is left and right $(T\text{-}S)_\Gamma$-subbiact on M , and since M is a right simple , N is a right $(T\text{-}S)_\Gamma$-subbiact implies that N = M . M .By the same we proof if M is a left simple $(T\text{-}S)_\Gamma$-biact .

Theorem 5.20 Let M be $(T\text{-}S)_\Gamma$-biact . Then
1. M is a right simple if and only if $m\Gamma S = M$ for all $m \in M$.
2. M is a left simple if and only if $T\Gamma m = M$ for all $m \in M$.

Theorem 5.22 Let M be $(T\text{-}S)_\Gamma$-biact . Then M is a simple $(T\text{-}S)_\Gamma$-biact if and only if $T\Gamma m\Gamma S = M$ for all $m \in M$.

Proof. $\Rightarrow$ it is clear
Conversely let N be $(T\text{-}S)_\Gamma$-subbiact ,and $n \in N$,implies $T\Gamma n\Gamma S = M$. Now let $m \in M = T\Gamma n\Gamma S$, then $m = t\alpha n\beta s \in N$, $t \in T$, $\alpha, \beta \in \Gamma$ and $s \in S$. Implies that M = N .

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