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RESEARCH ARTICLE

ANALYSIS AND COMPARATIVE STUDY OF NUMERICAL METHODS TO SOLVE ORDINARY DIFFERENTIAL EQUATION WITH INITIAL VALUE PROBLEM.

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Abstract

Since mathematics is a science of communication between us and the scientific sciences, in particular it introduces all the rules and problems as formulas, and searches for a solution. A part of the mathematics that is widely used in all sciences is the differential equation that will be studied in this dissertation. Each parts of these equations has its own method for solving, and we have generally studied the analytic methods in the calculus, and here we will introduce the numerical solutions. It is worth noting that in analytic methods cannot gives solution, for all equations this is where scientists have discovered the numerical methods that can be solved by those methods for those equations that are not solved in an analytical methods. In this Article, we first introduce differential equations and introduce a number of elementary topics for introduction so that the reader will get acquainted with these definitions and issues before the start of the process. Later in this article, one basic methods will be studied, the single step method, respectively, which relate to the initial value problem. Here we will examine in detail and analyze all the ways in which these methods are available. In the next step, all the good nesses, advantages, disadvantages that exist between these methods will be discussed, and also a comparative study we will have in this paper.

Introduction:-

Numerical analysis is a technique of doing high mathematical problems with a numerical process that is commonly used by scientists and engineers to resolve their difficulties. One of the main gains of numerical analysis is that a numerical reply can also be obtained when there is no "analytical" answer to the problem.

The outcome from numerical analysis is an approximation, which can be made closely as wanted. Analysis of errors in numerical methods is an essential part of the learning of numerical analysis [7]. Therefore, calculation of error is a requirement because it is a way of computing the efficiency of methods. Numerical methods require highly exhausting and monotonous calculate, which can be prepared only by using a computer. Numerical methods for ODEs are the procedures used to make numerical approximations for the solution of ODE.Since this paper is about solution of differential equation, so there is needed to have some basic concepts [9]. Differential equation is one of the essential, important parts of mathematics, which has significant applications, ∴ the equation which involves an

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unknown function and its derivatives of that unknown function known as differential equation, and classified in to two parts (1) ODE and (2) PDE.

ODE:
Ordinary differential equation is an equation in which unknown function depends only on one independent variable [5].

PDE:
If the unknown function depends on two or more independent variables, then the differential equation is a partial differential equation.

Numerical Methods
It is mathematical methods that are used to approximate the solution of complicated problems, and it can solve issues that cannot be solvable in the analysis method, and it is very useful because they are suitable for the use with computers [4].

Single step methods
The general form of this scheme is

$$y_{m+1} = y_m + \varphi(x_{m+1}, x_m, y_{m+1}, y_m; h) \qquad 1.1$$

The relation φ is called increment function and its function of $x_{m+1}, x_m, y_{m+1}, y_m; h$. If y_{m+1} can be solved simply by evaluating the RHS of (1.1) then the method is called explicit method, the general form is

$$y_{m+1} = y_m + \varphi(x_m, y_m; h) \qquad 1.2$$

If the RHS of (1.1) is dependent on y_{m+1}, then the method is called implicit method, which is not simply solvable. The condition which we use information, from only one preceding point is known as single step method, i.e to estimate y_{m+1}, we need previous point y_m only [8].

Initial value problem
Consider the differential equation $y - y' = 0$, we know that $y = ce^x$ is the general solution, now if we give a value of y for some x, then we will obtain the value of c For example, suppose $y = 2$ at $x = 0$ then we will obtain $c = 2$. If the order of equation is more than one, then we need more than one condition to obtain a unique solution. When all the conditions are defined at a Particular value of independent variable, then the problem is called initial value problem [3].

It should be noted that some authors have also done some research on the Numerical solution of ordinary differential equations, which we use as sources.

Akalu Abriham Anulo et al [1], is modified a new method based on a finite element method (correlation method) and also based on genetic algorithms. This method will be better in consecutive (single-step and multi-step) methods. In the article, only a comparative study is used. Gadamsetty revathi [2], has studied a relationship between probability issues and Runge kutta's method. In fact, the numerical method used in differential equations was studied by regression and covariance concept. Nikos E. Mastorakis [3]. This paper is a comparative study about two methods, namely Runge – Kutta and Fehlbarg method, in solution of differential equation, also in this paper completely error analysis studied, and also the methods illustrated by solving example and an application. Sankar Prasad Mondal et al [5].In this paper, the author proposes Runge-Kutta-Fehlberg's method for solving differential equations. This paper shows that this is a powerful mathematical tool for solving first-order linear differential equations in a fuzzy environment. Also discussed in this paper is the Runge-Kutta convergence. N. S HAWAGFEH et al [6]. In this paper, the author describes a new method known as Adomian de-composition. In fact, this article compares this method with Runge Kutta's method. The result of this study is that the new method is more precise and easy to apply. Md. Amirul Islam [7]. The author of this paper has discussed two basic methods, Euler's method and Runge-Kutta's method. In this article, both methods are examined in detail by providing examples, and they are given which one of the methods is better for each other than the numerical answer. Somayeh EZADI. [10]. This paper discusses the Taylor method and its relation with nonlinear optimization, and this paper first shows the ability of Neural Network and Taylor Network to approximate ODE solutions.

Analysis of Numerical Methods

Since defined previously numerical method is the rules of mathematics, which will find an approximate solution for a problem, so in this section we will introduce some Numerical method which is solved initial value problem

Taylor Series Method

This method is one of the most important and basic method for solving ODE with an IVP condition, and this method also validate some other methods. Now consider the general form of ODE with first order:

$$\frac{dy}{dx} = \varphi(x, y) \qquad 2.1$$

With initial value problem $y(x_0) = y_0$ such that $x \in [x_0, b]$, the differential equation (3.1) has a unique solution $y(x)$, on $[x_0, b]$, and $y(x)$ has continuous partial derivative of order say $p+1$ on $[x_0, b]$. Solution $y(x)$ of equation (3.1) can be expanded in a Taylor's series about any point, say $x = x_0$ as follows:

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \ldots + \frac{(x-x_0)^p}{2!}y^{(p)}(x_0) + \frac{(x-x_0)^{p+1}}{2!}y^{(p+1)}(\xi_m) \qquad 2.2$$

In equation (3.2) $x_0 < \xi_m < x$. Let $R_m = \frac{(x-x_0)^{p+1}}{2!}y^{(p+1)}(\xi_m)$, so the generalized form is:

$$y(x_{m+1}) = y(x_m) + h\,y'(x_m) + \frac{h^2}{2!}y''(x_m) + \ldots + \frac{h^p}{2!}y^{(p)}(x_m) + R_m \qquad 2.3$$

In here h is the step size i.e $x - x_0 = h$ or $x_{m+1} - x_m = h$.

In this method we should calculate the higher order derivative from initial value problem, and the $f(x, y)$ Must be differentiable as many times as we required, and the higher order derivatives are:

$$y'(x_m) = \varphi(x_m, y_m) \quad and \quad y''(x_m) = \left[\frac{\partial \varphi}{\partial x} + \varphi\frac{\partial \varphi}{\partial y}\right]_{(x_m, y_m)}$$

Now the Taylor series become:

$$y(x_{m+1}) = y(x_m) + h\varphi(x_m, y_m) + \frac{h^2}{2!}\left[\frac{\partial \varphi}{\partial x} + \varphi\frac{\partial \varphi}{\partial y}\right] + \cdots + \frac{h^p}{p!}\varphi^{(p)}(x_m, y_m) + \varepsilon_{p+1} \qquad 2.4$$

In equation (2.4) $\varepsilon_{p+1} = \frac{h^{p+1}}{(p+1)!}\varphi^{(p+1)}(x_m, y_m)$ is the Residual (Error).

Weakness of Taylor Series Method

This method is not applicable in all equations, because in this method we need the higher order derivation and receiving the higher derivatives of every relation is not easy. General form of HOD are defined as:

$$y' = \varphi(x, y) \quad , \quad y'' = \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = \varphi_x + \varphi \cdot \varphi_y \quad and$$

$$y''' = \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx}\right)\frac{dy}{dx} = \varphi_{xx} + 2\varphi_{xy} + \varphi^2\varphi_{yy} + \varphi_y(\varphi_x + \varphi \cdot \varphi_y)$$

The number of partial derivatives required, increases as the order of the derivatives of y increases. Therefore, we fined the computation of higher order derivatives are very difficult.

Concept of Error in Taylor Series Method

The concept of error in this method is due to truncation of the terms, if we calculate the series up to m^{th} terms derivative, the error will be of order $(x - x_0)^{m+1}$. If the amount of $(x - x_0)$ is large, the result may be unsatisfactory.

Improving Error Accuracy in Taylor series Method

We know that the error in Taylor method is, in the order of $(x-x_0)^{m+1}$.if $\left|x-x_0\right|$ is large, the error is also large. For improving the accuracy, we must divide the interval into subintervals $(x_0,x_1),(x_1,x_2),(x_2,x_3),....$ of equal length and computing $y(x_m)$,$m=1,2,...$ successively, using Taylor series expansion.

Euler's Method

Euler's Method is one of oldest numerical methods used for integrating the Ordinary Differential equation. Though this method is not used in practice, its understanding will help us to gain insight into nature of predictor – corrector method. Consider the differential equation of first order with the initial condition $y'=\varphi(x,y)$, $y(x_0)=y_0$ .The integral of this equation is a Curve in $xy-plane$. Hence, we find successively $y_1,y_2,\cdots y_m$ where y_m is the value of y at $x=x_m=x_0+mh$ where $m=1,2,3...$ and h being small. Here we use a property that in a small interval, a curve is nearly a straight line. Thus at (x_0,y_0), we approximate the curve by a tangent at that point. Therefore,

$$\left(\frac{dy}{dx}\right)_{(x_0,y_0)}=\frac{y-y_0}{x-x_0}=\varphi(x_0,y_0) \qquad 2.5$$

That is

$$y=y_0+(x-x_0)\ \varphi(x_0,y_0) \qquad 2.6$$

Therefore, the price of y equivalent to $x=x_1$ is known by:

$$y_1=y_0+(x-x_0)\varphi(x_0,y_0)=y_0+h\varphi(x_0,y_0) \qquad 2.7$$

The approximate solution curve in next interval $[x_1,x_2]$ with a straight line through the point (x_1,y_1) having its slope equivalent $\varphi(x_1,y_1)$, which is defined as:

$$y_2=y_1+h\varphi(x_1,y_1) \qquad 2.8$$

For every deferent intervals if continue the process, the general form will generate as:

$$y_{m+1}=y_m+h\varphi(x_m,y_m) \qquad 2.9$$

Error Analyze of Euler's Method

The concept of error in Euler Methods is belong to truncation error. If y_{m+1} is the exact solution of ordinary differential equation $y'=\varphi(x,y)$ with an initial value, and $\overline{y}_{m+1}$ is the approximate solution, then the difference between this two solution is called truncation error, let T_{m+1} is the truncation error then[1]:

$$T_{m+1}=y_{m+1}-\overline{y}_{m+1} \qquad 2.10$$

$$T_{m+1}=y_{m+1}-y_m-h\varphi(x_m,y_m),\ :\ \overline{y}_{m+1}=y_m+h\varphi(x_m,y_m) \qquad 2.11$$

$$T_{m+1}=\frac{h^2}{2!}y''(\xi_m):\ x_m<\xi_m<x_{m+1} \qquad 2.12$$

Hence, the truncation error in Euler Method of order h^2, also it must be said that, a round off error will be grows in this method, it happens by selecting the number of decimals in calculation.

Improving Error Accuracy in Euler's Methods

For improving the accuracy in this method, must decrease the magnitude of h that means we can decrease the length of step size. If we want to reduce the round off error, must select a greater number of decimal places in calculation.

Techniques of Runge–Kutta Family

Runge -Kutta "R-K" technique was organized by two German Scientist, "Runge" was around 1894 and after a few years was elaborated by "Kutta" and this technique is most famous since it is relatively accurate stable and easy. In case of solving ODE, whenever the calculation of higher derivation is complicated, this method is so useful, and also comparing with Euler simple method, it is more accurate [11]. The general form of s-stage of this family defined as:

$$y_{m+1} = y_m + hF(x_m, y_m; h) , \qquad i = 0,1,2,... \qquad 2.13$$

$$hF(x, y; h) = \sum_{r=1}^{s} w_r k_1 \qquad 2.14$$

$$k_1 = h\varphi(x, y) \qquad 2.15$$

$$k_r = h\varphi(x + \alpha_r h, y + \sum_{j=1}^{r-1} \beta_{rj} k_j), \quad r = 2,3,...,s \qquad 2.16$$

$$\alpha_r = \sum_{j=1}^{r-1} \beta_{rj} \qquad r = 2,3,...,s \qquad 2.17$$

We can also explicitly clarify the above formulas, so let $y_{i+1} = y_i + h\xi$ is the general form of equation then:

$$\xi = w_1 k_1 + w_2 k_2 + w_3 k_3 + \text{K} + w_m k_m \qquad 2.18$$

where

$$k_1 = \varphi(x_i, y_i) \qquad 2.19$$

$$k_2 = \varphi(x_i + \alpha_1 h,\ y_i + \beta_{11} k_1 h) \qquad 2.20$$

$$k_3 = \varphi(x_i + \alpha_2 h,\ y_i + \beta_{21} k_1 h + \beta_{22} k_2 h) \qquad 2.21$$

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$$k_m = \varphi(x_i + \alpha_{m-1} h,\ y_i + \beta_{m-1,1} k_1 h + \beta_{m-1,2} k_2 h + ... + \beta_{m-1,m-1} k_{m-1} h) \qquad 2.22$$

First Order R–K Techniques

As we know, for the various prices of s, this family will define different forms of formula. Now for S = 1, we will see that, this family will provide Euler's explicit formula.

$$y_{m+1} = y_m + h\varphi(x_m, y_m) \qquad m = 0,1,2,... \qquad 2.23$$

Second Order R–K Techniques

If we put S = 2 in general scheme, we can get a family of the second order as follows:

$$y_{m+1} = y_m + hF(x_m, y_m) \qquad i = 0,1,2,... \qquad 2.24$$

Where

$$hF(x, y; h) = (w_1 k_1 + w_2 k_2) \qquad 2.25$$

$$k_1 = h\varphi(x, y) \qquad 2.26$$

$$k_2 = h\varphi(x + \alpha_2 h, y + \beta_{21} k_1) \qquad 2.27$$

Now, we will determined the factors $w_1, w_2, \alpha_2, and\ \beta_{21}$. If $y(x)$ is exact answer and replaced in relation 2.24, the error of truncation is:

$$T_{m+1} = y_{m+1} - y_m - hF(x_m, y_m; h) \qquad 2.28$$

Will be fulfilled

$$T_{m+1} = O(h^3) \qquad 2.29$$

Now, from Equations 2.25 up to 2.27, we achieve

$$hF(x, y; h) = w_1 h\varphi(x, y) + w_2 h\varphi(x + \alpha_2 h, y + \beta_{21} k_1)$$

Developing the relation $\varphi(x+\alpha_2 h, y+\beta_{21}k_1)$ about the point (x, y), will achieve:

$$\varphi(x+\alpha_2 h, y+\beta_{21}k_1) = \varphi(x,y) + \alpha_2 h\varphi_x + \beta_{21}k_1\varphi_y + O(h^2)$$

Moreover $y(x+h) = y(x) + hy' + \frac{h^2}{2!}y'' + O(h^3) = y(x) + hy' + \frac{h^2}{2!}(\varphi_x + \varphi\cdot\varphi_y) + O(h^3)$

Then

$$T_{m+1} = \left[y(x+h) - y - hF(x,y;h)\right]_{(x_m, y_m)}$$

$$T_{m+1} = \left[y + hy' + \frac{h^2}{2!}(\varphi_x + \varphi\varphi_y) - y - w_1 h\varphi(x,y) - w_2 h\{\varphi(x,y) + \alpha_2 h\varphi_x + \beta_{21}k_1\varphi_y\} + O(h^3)\right]_{(x_m, y_m)}$$

$$T_{m+1} = \left[h\varphi(x,y) + \frac{h^2}{2!}(\varphi_x + \varphi\varphi_y) - w_1 h\varphi(x,y) - w_2 h\{\varphi(x,y) + \alpha_2 h\varphi_x + \beta_{21}h\varphi\varphi_y\} + O(h^3)\right]_{(x_m, y_m)}$$

Therefore, equation 3.44 requires that the coefficients must satisfy the following equations:

$$1 - w_1 - w_2 = 0 \quad \& \quad \frac{1}{2} - w_2\alpha_2 = 0 \quad \& \quad \frac{1}{2} - w_2\beta_{21} = 0$$

Therefore $w_1 = 1 - w_2 \quad \& \quad \alpha_2 = \beta_{21} = \frac{1}{2w_2} \quad \& \quad w_2 \neq 0$

Therefore, depending on the choice of w_2, there are generate a family of RK techniques which is all of order two. The most appropriate and satisfactory choice of w_2 is different values $w_2 = 1/2$ and 1 and $2/3$ which we define as:

Heun's Technique of order two by choosing $w_2 = 1/2$ and put this value in the given system, we will get $\alpha_1 = \beta_{11} = 1$ then we will obtain:

$y_{m+1} = y_m + \frac{h}{2}(k_1 + k_2)$ Where $k_1 = \varphi(x_m, y_m)$ *and* $k_2 = \varphi(x_m + h,\ y_m + k_1 h)$

This method is called Heun's technique.

Midpoint technique if we consider $w_2 = 1$, then we will obtain, $w_1 = 0$ and $\alpha_1 = \beta_{11} = 1/2$, then the required formula is:

$y_{m+1} = y_m + k_2 h$ Where $k_1 = \varphi(x_m, y_m)$ *and* $k_2 = \varphi(x_m + \frac{h}{2},\ y_m + k_1\frac{h}{2})$

This method is called Midpoint technique.

Ralston's technique for $w_2 = 2/3$ we will obtain, $w_1 = 1/3$ and $\alpha_1 = \beta_{11} = 3/4$, so the will achieved as:

$y_{m+1} = y_m + \frac{h}{3}(k_1 + 2k_2)$ Where $k_1 = \varphi(x_m, y_m)$ *and* $k_2 = \varphi(x_m + \frac{3}{4}h,\ y_m + \frac{3}{4}k_1 h)$

Which is called Ralston's Techniques.

Third Order R–K techniques for a price of S = 3 we can make a package of the following formulas

$$y_{m+1} = y_m + hF(x_m, y_m; h) \qquad m = 0,1,2,... \tag{2.30}$$

Therefore

$$hF(x,y;h) = w_1 k_1 + w_2 k_2 + w_3 k_3$$

$$k_1 = h\varphi(x,y) \;,\; k_2 = h\varphi(x+\alpha_2 h, y+\beta_{21}k_1) \;,\; k_3 = h\varphi(x+\alpha_3 h, y+\beta_{31}k_1+\beta_{32}k_2)$$

Now, the factors $w_1, w_2, w_3, \alpha_2, \alpha_3, \beta_{31}, \beta_{32}$ should be resolute. Now, growing by Taylor's series the value of k_2 and k_3 around the point (x, y), so we have:

$$k_2 = h\left[\varphi + \alpha_2 h\varphi_x + \beta_{21}k_1\varphi_y + \frac{1}{2}\left(\alpha_2^2 h^2\varphi_{xx} + 2\alpha_2 h\beta_{21}k_1\varphi_{xy} + \beta_{21}^2 k_1^2\varphi_{yy}\right) + O(h^3)\right]$$

$$= h\varphi + \alpha_2 h^2\varphi_x + \alpha_2 h^2\varphi\varphi_y + \frac{1}{2}\alpha_2^2 h^3(\varphi_{xx} + 2\varphi\varphi_{xy} + \varphi^2\varphi_{yy}) + O(h^4) = h\varphi + \alpha_2 h^2 F_1 + \frac{1}{2}\alpha_2^2 h^3 F_2 + O(h^4)$$

Where $F_1 = \varphi_x + \varphi\varphi_y$ & $F_2 = \varphi_{xx} + 2\varphi\varphi_{xy} + \varphi^2\varphi_{yy}$

$$k_3 = h\left[\varphi + \alpha_3 h\varphi_x + (\beta_{31}k_1 + \beta_{32}k_2)\varphi_y + \frac{1}{2}\left(\alpha_3^2 h^2\varphi_{xx} + 2\alpha_3 h(\beta_{31}k_1 + \beta_{32}k_2)\varphi_{xy} + (\beta_{31}k_1 + \beta_{32}k_2)^2\varphi_{yy}\right) + O(h^3)\right]$$

$$= h\varphi + \alpha_3 h^2\varphi_x + h\left[(\alpha_3 - \beta_{32})k_1 + \beta_{32}k_2\right]\varphi_{yy}$$

$$+ \frac{h}{2}\left[\alpha_3^2 h^2\varphi_{xx} + 2\alpha_3 h[(\alpha_3 - \beta_{32})k_1 + \beta_{32}k_2]\varphi_{xy} + [(\alpha_3 - \beta_{32})k_1 + \beta_{32}k_2]^2\varphi_{yy}\right] + O(h^4)$$

$$k_3 = h\varphi + h^2\alpha_3 F_1 + h^3\left(\alpha_3\beta_{32}F_1\varphi_y + \frac{1}{2}\alpha_3^2 F_2\right) + O(h^4)$$

Replacing k_1, k_2 and k_3, in $hF(x,y;h) = w_1k_1 + w_2k_2 + w_3k_3$, we will find:

$$hF(x,y;h) = w_1 h\varphi + w_2\left(h\varphi + \alpha_2 h^2 F_1 + \frac{1}{2}\alpha_2^2 h^3 F_2\right) + w_2\left[h\varphi + h^2\alpha_3 F_1 + h^3(\alpha_3\beta_{32}F_1\varphi_y + \frac{1}{2}\alpha_3^2 F_2\right] + O(h^4)$$

This implies

$$F(x,y;h) = w_1\varphi + w_2\left(\varphi + \alpha_2 hF_1 + \frac{1}{2}\alpha_2^2 h^2 F_2\right) + w_2\left[\varphi + h\alpha_3 F_1 + h^2(\alpha_3\beta_{32}F_1\varphi_y + \frac{1}{2}\alpha_3^2 F_2\right] + O(h^3) \quad 2.31$$

If $y(x)$ which is the exact answer is satisfies the relation 2.30, then:

$$y(x+h) = y(x) + hF(x, y(x); h) \quad 2.32$$

Again,

$$y(x+h) = y(x) + hy' + \frac{h^2}{2!}y'' + \frac{h^3}{3!}y''' + O(h^4)$$

$$= y(x) + h\varphi + \frac{h^2}{2!}\left(\varphi_x + \varphi\varphi_y\right) + \frac{h^3}{3!}\left[(\varphi_x + \varphi\varphi_y)\varphi_y + \varphi_{xx} + 2\varphi\varphi_{xy} + \varphi^2\varphi_{yy}\right] + O(h^4)$$

Therefore,

$$F(x,y;h) = \frac{y(x+h) - y(x)}{h} = \varphi + \frac{h}{2!}F_1 + \frac{h^2}{3!}\left[F_1\varphi_y + F_2\right] + O(h^3) \quad 2.33$$

Now, matching relations "2.31" and "2.33" displays that

$$w_1 + w_2 + w_3 = 1 \quad \& \quad w_2\alpha_2 + w_3\alpha_3 = \frac{1}{2} \quad \& \quad \frac{1}{2}w_2\alpha_2^2 F_2 + w_3\alpha_3\beta_{32}F_1 f_y + \frac{1}{2}w_3\alpha_3^2 F_2 = \frac{1}{6}\left[F_1 f_y + F_2\right]$$

Above identity is done

$$w_2\alpha_2^2 + w_3\alpha_3^2 = \frac{1}{3} \quad \& \quad w_3\alpha_3\beta_{32} = \frac{1}{6}$$

Hence, we achieve the scheme of blow relations:

$$w_1 + w_2 + w_3 = 1 \quad \& \quad w_2\alpha_2 + w_3\alpha_3 = \frac{1}{2} \quad \& \quad w_2\alpha_2^2 + w_3\alpha_3^2 = \frac{1}{3} \quad \& \quad w_3\alpha_3\beta_{32} = \frac{1}{6}$$

This system is solved from four equations for six $w_1, w_2, w_3, \alpha_2, \alpha_3,$ & β_{32} unknowns, in this sense we will get two-parameter of the three order R–K schemes.

Heun's technique of order three:

In this technique we consider the below values of unknowns to achieved a new formula.

$$w_1 = \frac{1}{4}, w_2 = 0,\ w_3 = \frac{3}{4},\ \alpha_2 = \frac{1}{3}, \alpha_3 = \frac{2}{3},\ \textit{and}\ \ \beta_{32} = \frac{2}{3}$$

We will receive the resulting recursive formulation

$$y_{m+1} = y_m + \frac{h}{4}(k_1 + 3k_3), \quad m = 0,1,2... \qquad 2.34$$

Where

$$k_1 = \varphi(x_m, y_m) \quad \& \quad k_2 = \varphi(x_m, y_m + \frac{1}{3}k_1) \quad \& \quad k_3 = \varphi(x_m + \frac{2}{3}h,\ y_m + \frac{2}{3}k_2)$$

Usual R–K technique of 3thorder: by using blew value this technique will be realized:

$$w_1 = \frac{1}{6}, w_2 = \frac{2}{3}, w_3 = \frac{1}{6}, \alpha_2 = \frac{1}{2}, \alpha_3 = 1, \ and \ \beta_{32} = 2$$

Achieved the following recursive formula

$$y_{m+1} = y_m + \frac{h}{6}(k_1 + 4k_2 + k_3) \qquad 2.35$$

Where $k_1 = \varphi(x_m, y_m)$, $k_2 = \varphi(x_m + \frac{1}{2}h\ ,\ y_m + \frac{1}{2}k_1)$, $k_3 = \varphi(x_m + h\ ,\ y_m - k_1 + 2k_2)$

Fourth Order R–K techniques

By same process, which we done no second and third order R-K methods we will find thirteen unknown with eleven equations [4]. So, there are accrue two arbitrary parameters, this two parameter chosen freely, one of the most important value can be chosen like blew, which generate the fourth order RK technique.

$$w_1 = w_4 = \frac{1}{6}, w_2 = w_3 = \frac{1}{3}\ ,\ \alpha_2 = \alpha_3 = \frac{1}{2}\ ,\ \alpha_4 = 1,\ \beta_{21} = \frac{1}{2},\ \beta_{31} = 0,\ \beta_{32} = \frac{1}{2},\ \beta_{41} = 0,\ \beta_{42} = 0, \ and \ \beta_{43} = 1$$

Therefore, we get the following fourth order R–K formula, which has been considered as usual R–K technique.

$$y_{m+1} = y_m + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad m = 0,1,2,... \qquad 2.36$$

Where $k_1 = \varphi(x_m, y_m)$, $k_2 = \varphi\left(x_m + \frac{h}{2}, y_m + \frac{k_1}{2}\right)$, $k_3 = \varphi\left(x_m + \frac{h}{2}, y_m + \frac{k_2}{2}\right)$ & $k_4 = \varphi(x_m + h, y_m + k_3)$

Obviously, the truncation error of this technique is $O(h^5)$.

Error Approximation for Runge Kutta Technique

One of the serious reasons Runge Kutta's scheme is to estimate the error. For all single step schemes like Runge Kutta Technique, we explain a local truncation error as the terms we discard when making a numerical arrangement from Taylor series expansion. We have noted that the truncation error in p^{th} order Runge Kutta technique is Ch^{p+1}, where C is some constant. Limits on C for $p = 1, 2, 3...$ also exist. The derivation of these limits is not a simple problem since they need some amounts in evaluating them. Under the localizing supposition that no earlier errors have been made, we could write

$$T_{m+1} = y(x_{m+1}) - y_{m+1} = \mu(x_m, y(x_m))h^{p+1} + O(h^{p+2}) \qquad 2.37$$

Here p shows the order of the RK technique, $\mu(x_m, y(x_m))h^{p+1}$ is the base of local truncation error.

Next we will calculate y^*_{m+1}, a second estimate to $y(x_{m+1})$, achieved by applying the same process at x_{m-1} with step distance $2h$. Then we will obtain

$$T_{m+1} = y(x_{m+1}) - y^*_{m+1} = \mu(x_{m-1}, y(x_{m-1}))(2h)^{p+1} + O(h^{p+2}) \qquad 2.38$$

After some Simplification we will obtain:

$$\mu(x_m, y(x_m))(h)^{p+1} = T_{m+1} = \frac{y(x_{m+1}) - y^*_{m+1}}{2^{p+1} - 1} \qquad 2.39$$

Equation (2.39) is a means to obtain quick approximations of local truncation errors in calculations using the S-stage Runge Kutta technique without h obtaining an exact answer first [12].

Convergence of Numerical Methods

Consider the general form of single step methods

$y_{m+1} = y_m + hF(x_m, y_m, f(x_m, y_m), h) + \varepsilon_{m+1}$ 2.40

Let $\bar{y}_{m+1} = \bar{y}_m + hF\left(x_m, \bar{y}_m, f(x_m, \bar{y}_m)\right)$ satisfied the scheme (083) in exact since. A method is said to be convergence if $\forall\ n = 0,1,2,\ ...\ N_m$ we have $\left|\bar{y}_m - y_m\right| \le C(h)$ where $C(h)$ is an infinitesimal with respect to h. Further the method is said to be convergent with order p if $\exists\ d > 0$ such that $C(h) \approx dh^{p+1}$ [5].

Stability of Numerical methods

In each IVP, we need answer to $x > x_0$ and generally up to a unite x = b. the most important issue for methods of numerical in an IVP is step size, the step size in an IVP must be Chosen accurately. All the calculations have two types of errors namely round-off error and truncation error. By choosing higher order methods one can control the Truncation error, but the Round-off errors in a calculation is not controllable they can develop and lastly destroy the true answer, in this cases the method is called unstable. Unstable case create whenever step size is chosen greater than the acceptable limit cost. There exist a limitations to step size in explicit methods that can be used, but not exist any limit for step size in many implicit methods that can be used, Likewise methods are named unconditionally stable methods [6].

The linearized scheme of the IVP is given by $y' = \mu y$ where, $\mu < 0$ and $y(x_0) = y_0$. The scheme of single step method used in this differential equation to get the difference equation $y_{m+1} = E(\mu h) y_m$, where $E(\mu h)$ is named amplification element. If $\left|E(\mu h) < 1\right|$, then round-off and all other errors decrease and the technique gives convergent answers and we say that the technique is stable. This form gives a bound on the step size h which can be used in the calculations. Now, will introduced some following stability conditions to some single step methods:

1. In numerical methods of Euler $-2 < \mu h < 0$.
2. In second order numerical methods of RK technique $-2 < \mu h < 0$
3. In Classical fourth order methods of RK technique $-2.78 < \mu h < 0$.

Comparative Study and Results

In this chapter firstly will describe a comparative study and will mentioned all Algorithms of numerical methods that defined in this dissertation, for compare these methods we will consider an example and solve it with different step sizes, and for every step size and errors we will mention tables, also with use of Excel we will draw the bar graph of error relation.

Example1. Consider the IVP and find the value of y at $x = 0.5$ whenever, $y' = 4e^{0.8x} - 0.5y$ and $y(0) = 2$. The exact solution of this problem is $y = 2/1.3(e^{0.8x} - e^{-0.5}) + 2e^{-0.5x}$, the approximate result and errors are given in below tables, this example solved with three different step size and arranged in below tables, table A is arranged with $h = 0.1$, and table A' is shows the Max error, similarly, Table $B \& C$ shows the solution with $h = 0.05$ and $h = 0.025$, and table $B' \& C'$ shows Max errors.

x	Exact Answer	Euler Method	Heun's Order 2	Midpoint Method	Ralston Method	Heun's Order3	R-K Order 3	R-K Order 4
0.0	**2.0000**	**2.00000**	**2.00000**	**2.00000**	**2.00000**	**2.00000**	**2.00000**	**2.00000**
0.1	**2.3088**	**2.30000**	**2.31591**	**2.31557**	**2.31574**	**2.30865**	**2.30879**	**2.30879**
0.2	**2.6364**	**2.61831**	**2.65068**	**2.65000**	**2.65034**	**2.63608**	**2.63636**	**2.63636**
0.3	**2.9846**	**2.95680**	**3.00625**	**3.00522**	**3.00573**	**2.98419**	**2.98461**	**2.98462**
0.4	**3.3556**	**3.31746**	**3.38472**	**3.38332**	**3.38401**	**3.35502**	**3.35561**	**3.35561**
0.5	**3.7515**	**3.70244**	**3.78832**	**3.78653**	**3.78742**	**3.75077**	**3.75151**	**3.75152**

Table1 :-Error table by five step

x	Exact Answer	Euler Method	Heun's Order 2	Midpoint Method	Ralston Method	Heun's Order3	R-K Order 3	R-K Order 4
0.0	**2.0000**	**0.00000**	**0.0000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**
0.1	**2.3088**	**0.00880**	**0.0071**	**0.00677**	**0.00694**	**0.00015**	**0.00001**	**0.00001**

0.2	**2.6364**	**0.01809**	**0.0143**	**0.01360**	**0.01394**	**0.00032**	**0.00004**	**0.00004**
0.3	**2.9846**	**0.00278**	**0.0216**	**0.02062**	**0.02113**	**0.00041**	**0.00001**	**0.00002**
0.4	**3.3556**	**0.03814**	**0.0291**	**0.02772**	**0.02841**	**0.00058**	**0.00001**	**0.00001**
0.5	**3.7515**	**0.04906**	**0.0368**	**0.03503**	**0.03592**	**0.00073**	**0.00001**	**0.00002**

TABLE A'

Result1.

In the table A', error is not so accurate with $h = 0.1$, Euler, Euler Modified, Heun's of order 2, Midpoint Method, Ralston Method which is called also the first and second order Runge Kutta method haven't a good accuracy but Heun's of order 3, R K of order 3 and R K of order 4, have a good accuracy for solving.

x	Exact Answer	Euler Method	Heun's Order 2	Midpoint Method	Ralston Method	Heun's Order3	R-K Order 3	R-K Order 4
0.0	**2.0000**	**2.00000**	**2.00000**	**2.00000**	**2.00000**	**2.00000**	**2.00000**	**2.00000**
0.05	**2.1522**	**2.15000**	**2.15399**	**2.15395**	**2.15397**	**2.15214**	**2.15216**	**2.15216**
0.10	**2.3088**	**2.30441**	**2.31245**	**2.31237**	**2.31241**	**2.30876**	**2.30879**	**2.30879**
0.15	**2.4701**	**2.46346**	**2.47562**	**2.47549**	**2.47556**	**2.47006**	**2.47011**	**2.47011**
0.20	**2.6364**	**2.62737**	**2.64373**	**2.64356**	**2.64364**	**2.63629**	**2.63636**	**2.63636**
0.25	**2.8078**	**2.79639**	**2.81702**	**2.81681**	**2.81691**	**2.80769**	**2.80778**	**2.80778**
0.30	**2.9846**	**2.97076**	**2.99575**	**2.99550**	**2.99562**	**2.98451**	**2.98462**	**2.98462**
0.35	**3.1671**	**3.15074**	**3.18018**	**3.17988**	**3.18003**	**3.16701**	**3.16714**	**3.16714**
0.40	**3.3556**	**3.33660**	**3.37059**	**3.37024**	**3.37041**	**3.35546**	**3.35560**	**3.35561**
0.45	**3.5503**	**3.52861**	**3.56725**	**3.56686**	**3.56705**	**3.55014**	**3.55030**	**3.55030**
0.50	**3.7515**	**3.72706**	**3.77046**	**3.77002**	**3.77024**	**3.75134**	**3.75152**	**3.75152**

Table 2:-Error table by ten steps

x	Exact Answer	Euler Method	Heun's Order 2	Midpoint Method	Ralston Method	Heun's Order3	R-K Order 3	R-K Order 4
0.0	**2.0000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**
0.05	**2.1522**	**0.00220**	**0.00179**	**0.00175**	**0.00177**	**0.00006**	**0.00004**	**0.00004**
0.10	**2.3088**	**0.00439**	**0.00365**	**0.00357**	**0.00361**	**0.00004**	**0.00001**	**0.00001**
0.15	**2.4701**	**0.00664**	**0.00552**	**0.00539**	**0.00546**	**0.00004**	**0.00001**	**0.00001**
0.20	**2.6364**	**0.00903**	**0.00733**	**0.00716**	**0.00724**	**0.00011**	**0.00004**	**0.00004**
0.25	**2.8078**	**0.01141**	**0.00922**	**0.00901**	**0.00911**	**0.00011**	**0.00002**	**0.00002**
0.30	**2.9846**	**0.01384**	**0.01115**	**0.0109**	**0.01102**	**0.00009**	**0.00002**	**0.00002**
0.35	**3.1671**	**0.01636**	**0.01308**	**0.01278**	**0.01293**	**0.00009**	**0.00004**	**0.00004**
0.40	**3.3556**	**0.01900**	**0.01499**	**0.01464**	**0.01481**	**0.00014**	**0.00000**	**0.00001**
0.45	**3.5503**	**0.02169**	**0.01695**	**0.01656**	**0.01675**	**0.00016**	**0.00000**	**0.00000**
0.50	**3.7515**	**0.02444**	**0.01896**	**0.01852**	**0.01852**	**0.00016**	**0.00002**	**0.00002**

TABLE B'

Result2.

Consider the average Error of table A' and table B' given that:

h	Euler average Error	Heun's 2 average Error	Midpoint average Error	Ralston average Error	Heun's 3 average Error	R-K 3 average Error	R-K 4 average Error
0.1	**0.07180**	**0.02178**	**0.02075**	**0.02127**	**0.000438**	**0.000016**	**0.000020**
0.05	**0.01290**	**0.01026**	**0.010028**	**0.010122**	**0.000100**	**0.0000020**	**0.000021**

By comparing the average Error, clearly we see that for step size 0.05 more accurate than $h = 0.1$.

Now by using MATLAB, we will plot the above Example in $0 \le x \le 5$, since in small interval the Errors will not realizable, also by zooming the figure 1 in a small interval, it will be more realizable to consider the accuracy of these methods.

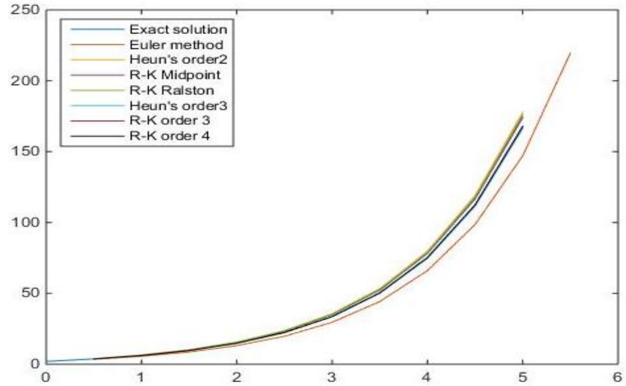

Figure 1

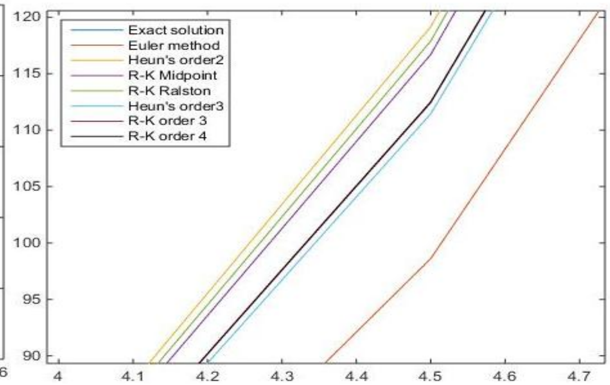

Figure 2

Result4.
By considering the figure 1 it is Obvious only the Euler's method is not more accurate and it has large error than others method. But the other methods are relatively have more accuracy.

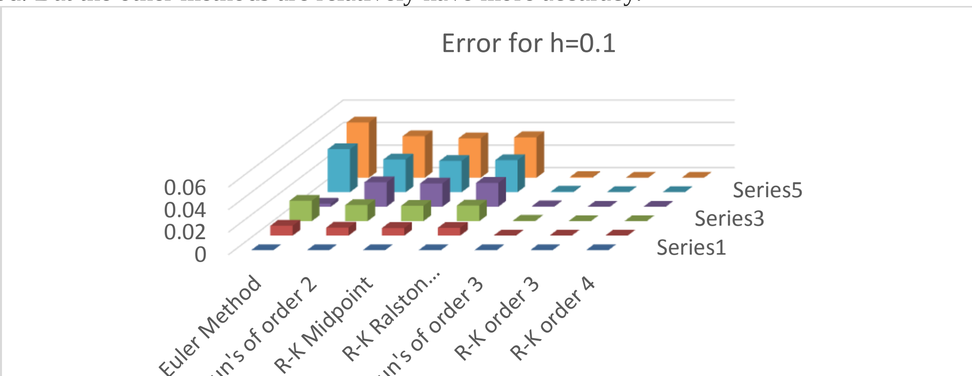

Figure 3

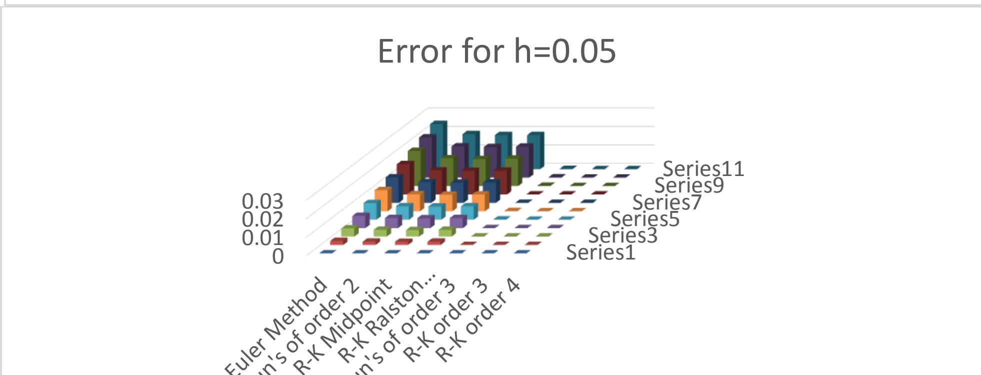

Figure 4

Result5.
Consider the figure3, it is a bar graph for showing the Error with h=0.1, it shows that Euler method, Heun's order 2, Midpoint and Ralston method errors are not accurate it lies, from zero up to approximately 0.05, but the last three method is more than accurate from all methods. Figure 4 shows the errors in ten steps, it means that by taking h=0.05, the methods became more accurate. Heun's of order 2, Midpoint and Ralston method lies between 0 and 0.02, but from all these methods, the Heun's order3, R-K order3 and R-K order 4 is the best methods, since these are more accurate and error is approximately zero.

Conclusion:-

In this report single step methods of IVP from ODE has been studied. Various methods such as Taylor's series method, Euler's method, Runge Kutta first, second, third, fourth order has been described. These methods are critically studied and differentiated. Comparative study of these methods has been done using different step size with use of MATLAB. Various table, graphs and bar graph depicts the more convergent methods among all off these mentioned methods.

As we know that, for small step size, number of iteration are large that gives more accurate result by this, we concluded that R-K method of fourth order is the best method, to give accurate result, and Euler method gives least accurate result. R-K method of fourth order gives less error while Euler's method gives more error which is cleared in the comparative table, graph and bar graph in the report.

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