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RESEARCH ARTICLE

QUASI-INJECTIVE GAMMA MODULES.

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Abstract

In this paper we introduce the concept of quasi-injective gamma modules as a proper generalization of injective gamma modules and quasi-injective modules. An $R_\Gamma -$ module M is called quasi-injective if for any $R_\Gamma -$submodule A of M and $R_\Gamma -$ homomorphism f from A to M there is an $R_\Gamma -$endomorphism of M which extends f. We are extending some results from module theory to gamma module theory, we established that every $R_\Gamma -$ module has quasi-injective hull which is unique up to isomorphism. Morever, if M is quasi-injective, then $\mathrm{End}_{R_\Gamma}(M)\,/J(\mathrm{End}_{R_\Gamma}(M))$ is regular $\Gamma -$ ring.

1. Introduction

The notion of $\Gamma -$ring was first introduced by N. Nobusawa [9] and then Barnes [3] generalized the definition of Nobusawa's gamma rings. R. Ameri and R. Sadeghi [2] studied gamma module, gamma submodule, homomorphism of gamma modules. They obtained some basic results of gamma modules. The authors in [1] introduced and studied the concept of injective gamma module, divisible gamma module and essential gamma submodule. They proved that every gamma module can be embedded in injective gamma module.

In this paper, we introduce and study the concept of quasi-injective gamma modules as a proper generalization of injective gamma modules and quasi-injective modules.

2. Preliminaries

Let R and Γ be two additive abelian groups, R is called a $\Gamma -$ ring (in the sense of Barnes), if there exists a mapping $\cdot: R \times \Gamma \times R \longrightarrow R$, written $\cdot (r,\gamma,s) \longmapsto r\gamma s$ such that $(a+b)\alpha c = a\alpha c + b\alpha c$ and $a(\alpha+\beta)c = a\alpha c + a\beta c$, $a\alpha(b+c) = a\alpha c + b\alpha c$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a,b,c \in R$ and $\alpha,\beta \in \Gamma$ [3]. A subset A of $\Gamma -$ring R is said to be a left(right) ideal of R if A is an additive subgroup of R and $R\Gamma A \subseteq A$ ($A\Gamma R \subseteq A$), where $R\Gamma A = \{r\alpha a : r \in R, \alpha \in \Gamma, a \in A\}$. If A is both right and left ideal, we say that A is an ideal of R [3]. An element 1 in $\Gamma -$ ring R is unity if there exists element, say 1 in R and $\gamma_\circ \in \Gamma$ such that $r = 1\gamma_\circ r = r\gamma_\circ 1$ for every $r \in R$, unities in $\Gamma -$ rings differ from unities in rings, it is possible for a $\Gamma -$ ring have more than one unity [7].

Let R be a $\Gamma -$ring and M be an additive abelian group. Then M together with a mapping $\cdot: R \times \Gamma \times M \rightarrow M$,written $\cdot (r,\gamma,m) \longmapsto r\gamma m$ such that

1- $r\alpha(m_1 + m_2) = r\alpha m_1 + r\alpha m_2$
2- $(r_1 + r_2)\alpha m = r_1\alpha m + r_2\alpha m$
3- $r(\alpha+\beta)m = r\alpha m + r\beta m$
4- $(r_1\alpha r_2)\beta m = r_1\alpha(r_2\beta m)$

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for each $r, r_1, r_2 \in R$, $\alpha, \beta \in \Gamma$ and $m, m_1, m_2 \in M$, is called a left R_Γ −module, similarly one can defined right R_Γ −module [2]. A left R_Γ −module M is unitary if there exist elements, say 1 in R and $\gamma_\circ \in \Gamma$ such that $1\gamma_\circ m = m$ for every $m \in M$.

Let M be an $R_\Gamma -$ module. Anonempty subset N of M is said to be an R_Γ −submodule of M (denoted by $N \leq M$) if N is a subgroup of M and $R\Gamma N \subseteq N$, where $R\Gamma N = \{r\alpha n : r \in R, \alpha \in \Gamma, n \in N\}$ [2]. An $R_\Gamma -$ module M is called simple if $R\Gamma M \neq 0$ and the only $R_\Gamma -$ submodules of M are M and 0 [4]. If X is a nonempty subset of M, then the $R_\Gamma -$ submodule of M generated by X denoted by $\langle X \rangle$ and $\langle X \rangle = \cap \{N \leq M : X \subseteq N\}$, X is called the generator of $\langle X \rangle$ and $\langle X \rangle$ is finitely generated if $|X| < \infty$. If $X = \{x_1, \dots, x_n\}$, then $\langle X \rangle = \{\sum_{i=1}^{m} n_i x_i + \sum_{j=1}^{k} r_j \gamma_j x_j : k, m \in \mathbb{N}, n_i \in \mathbb{Z}, \gamma_j \in \Gamma, r_j \in R, x_i, x_j \in X\}$. In particular, if $X = \{x\}$, then $\langle X \rangle$ is called the cyclic $R_\Gamma -$ submodule of M generated by x. If M is unitary, then $\langle X \rangle = \left\{\sum_{i=1}^{n} r_i \gamma_i x_i : n \in \mathbb{N}, \gamma_i \in \Gamma, r_i \in R, x_i \in X\right\}$ [2].

Let M and N be two $R_\Gamma -$ modules. A mapping $f: M \to N$ is called homomorphism of $R_\Gamma -$ modules (simply $R_\Gamma -$ homomorphism) if $f(x + y) = f(x) + f(y)$ and $f(r\gamma x) = r\gamma f(x)$ for each $x, y \in M, r \in R$ and $\gamma \in \Gamma$. An $R_\Gamma -$ homomorphism is $R_\Gamma -$ monomorpism if it is one-to-one and $R_\Gamma -$ epimorphism if it is onto, the set of all $R_\Gamma -$ homomorphisms from M into N denote by $\mathrm{Hom}_{R_\Gamma}(M, N)$. In particular, if $M = N$, then $\mathrm{Hom}_{R_\Gamma}(M, N)$ denote by $\mathrm{End}_{R_\Gamma}(M)$. If M is $R_\Gamma -$ module, then $\mathrm{End}_{R_\Gamma}(M)$ is a $\Gamma -$ ring with the mapping $\cdot : \mathrm{End}_{R_\Gamma}(M) \times \Gamma \times \mathrm{End}_{R_\Gamma}(M) \to \mathrm{End}_{R_\Gamma}(M)$ denoted by $\cdot (f, \gamma, g) \mapsto f\gamma g$ where $f\gamma g(x) = f(g(1\gamma x))$, for $f, g \in \mathrm{End}_{R_\Gamma}(M)$, $\gamma \in \Gamma$ and $x \in M$. All modules in this paper are unitary left R_Γ −modules and $\gamma_\circ \in \Gamma$ denote to the element such that $1\gamma_\circ$ is the unity, in this case M is a right $\mathrm{End}_{R_\Gamma}(M)$ −module with the mapping $\cdot : M \times \Gamma \times \mathrm{End}_{R_\Gamma}(M) \to \mathrm{End}_{R_\Gamma}(M)$ by $\cdot (x, \gamma, f) \mapsto x\gamma f$ where $x\gamma f = f(1\gamma x)$, for $f \in \mathrm{End}_{R_\Gamma}(M), \gamma \in \Gamma$ and $x \in M$ [2].

Let M and N be two $R_\Gamma -$ modules. Then M is called N −injective if for any $R_\Gamma -$ submodule A of N and $R_\Gamma -$ homomorphism $f: A \to M$, there exists an $R_\Gamma -$ homomorphism $g: N \to M$ such that $gi = f$ where i is the inclusion mapping. An $R_\Gamma -$ module M is injective if it is $N -$ injective for any $R_\Gamma -$ module N [1]. An $R_\Gamma -$ submodule N of $R_\Gamma -$ module M is essential (denote by $N \leq_e M$) if every nonzero $R_\Gamma -$ submodule of M has nonzero intersection with N, in this case we say that M is an essential extension of N [1]. It is proved in [1], that every gamma module can be embedded in injective gamma module. The minimal injective extension of M is called injective hull (denote by $E(M)$) which is unique up to isomorphism.

3. Quasi-injective gamma module

In this section we introduce the concept of quasi-injective gamma modules as a generalization of injective gamma modules.

Definition 3.1 An $R_\Gamma -$ submodule N of $R_\Gamma -$ module M is called direct summand if there exists an $R_\Gamma -$ submodule K of M such that $M = N + K$ and $N \cap K = 0$, in this case M is written as $M = N \oplus K$.
The $R_\Gamma -$ submodules 0 and M are always direct summand of M.

Definition 3.2 Let N be an $R_\Gamma -$ submodule of an R_Γ −module M. A complement of N in M is any $R_\Gamma -$ submodule denoted by N^c of M which is maximal with respect to the property $N \cap N^c = 0$.
By Zorn's lemma, one can be show that every submodule of gamma module has a complement submodule which is not unique in general.

Definition 3.3 An $R_\Gamma -$ submodule A of an $R_\Gamma -$ module M is called closed in M if it has no proper essential extension in M, that is, the only solution of the relation $A \leq_e K \leq M$ is $A = K$. It is easy to see that every direct summand of M is closed.

In this lemma, we see that every $R_\Gamma -$ submoduleis a direct summand of an essential $R_\Gamma -$ submodule.

Lemma 3.4 Let N be an $R_\Gamma -$ submodule of an R_Γ −module M. Then $N \oplus N^c \leq_e M$.
Proof. For each $R_\Gamma -$ submodule K of M such that $K \cap (N \oplus N^c) = 0$, if $a \in N \cap (N^c \oplus K)$, then $a = b + k$ where $b \in N^c$ and $k \in K$, so $k = a - b \in K \cap (N \oplus N^c) = 0$, hence $a = b \in N \cap N^c = 0$, so $N \cap (N^c \oplus K) = 0$, by maximality of N^c we have $N^c = N^c \oplus K$, so $K = 0$, hence $N + N^c \leq_e M$.

Lemma 3.5 Let N be $R_\Gamma -$ submodule of an $R_\Gamma -$ module M and K a complement of N in M . Then:
1. There exists a complement L of K in M such that $N \leq L$.

2. L is a maximal essential extension of N.
3. If N is closed, then $N = L$.

Proof.

1. By Zorn's lemma there exists a complement L of K which contains N.
2. For any $A \leq L$, since $L \cap K = 0$, then $A \cap K = 0$. Let $0 \neq x = a + k \in (A + K) \cap N$ where $a \in A$ and $k \in K$. Then $k = x - a \in K \cap L = 0$, so $k = 0$ and $x = a \in N \cap A$, hence $N \cap A \neq 0$, so $N \leq_e L$, if P is $R_\Gamma -$ submodule of M contains L properly, then $P \cap K \neq 0$ and $(P \cap K) \cap N = P \cap (K \cap N) = P \cap 0 = 0$, thus P is not essential extension of N.
3. Follows from (2).

Definition 3.6 An $R_\Gamma -$ module M is called quasi-injective if for any $R_\Gamma -$ submodule A of Q and for any $R_\Gamma -$ homomorphism $f: A \to M$ there exists an $R_\Gamma -$endomorphism g of M such that $gi = f$ where i is the inclusion mapping of A into M.
In fact, M is a quasi-injective if and only if M is $M -$ injective [1].

The proof of the following propositions follow from proposition(3.13) in [1].

Proposition 3.7 An $R_\Gamma -$ module M is quasi-injective if $f(M) \subseteq M$ for every $f \in \mathrm{End}(E(M))$.

Corollary 3.8 Let M be a quasi-injective $R_\Gamma -$ module and $\{A_\lambda : \lambda \in \Lambda\}$ be a family of an independent set of $R_\Gamma -$ submodules of M, then $M \cap (\oplus_{\lambda\in\Lambda} A_\lambda) = \oplus_{\lambda\in\Lambda}(M \cap A_\lambda)$.

Corollary 3.9 Let M be a quasi-injective $R_\Gamma -$module , then:

1. Every $R_\Gamma -$ submodule of M is essential in a direct summand of M.
2. If an $R_\Gamma -$ submodule N of M isomorphic to a summand of M, then N is a summand of M.

Proof.

1. Assume $N \leq M$ and $E(M) = E_1 \oplus E_2$ where $E_1 = E(N)$, by proposition(3.7) $M = (M \cap E_1) \oplus (M \cap E_2)$, by lemma(3.3) in [1], $N \leq_e M \cap E_1$.
2. Assume $N \cong K$ and K is a direct summand of M , then there exists $R_\Gamma -$ submodule K_1 of M such that $M = K \oplus K_1$ and $R_\Gamma -$ isomorphism $\alpha: N \to K$, frome [1] K is $M -$ injective, so α can be extended to an $R_\Gamma -$ homomorphism $\beta: M \to K$ such that $\alpha = \beta i$ where i is the inclusion mapping, so $M = \mathrm{Im}(i) \oplus \mathrm{Ker}(\beta)$, hence N is a summand of M.

Proposition 3.10 Let N be a closed $R_\Gamma -$ submodule of an $R_\Gamma -$module M. If M is quasi-injective, then N is $M -$ injective.

Proof. Let K is $R_\Gamma -$ submodule of M and $f: K \to N$ is an $R_\Gamma -$homomorphism, define $\Omega = \{(K', f'): K \leq K' \leq M, f'$ extended of f to $K'\}$ by Zorn's lemma Ω has a maximal element $(K_\circ, f_\circ)$, since M is quasi-injective, then $f_\circ$ can extended to an $R_\Gamma -$homomorphism $g: M \to M$. If $g(M) \nsubseteq N$, let L be a complement of N in M, since N closed, then N is complement of L , since $N \subset N + g(M)$, so $[N + g(M)] \cap L \neq 0$, let $0 \neq x = a + b$ where $a \in N$ and $b \in g(M)$, if $b \in N$, then $x = a + b \in N \cap L = 0$ contradiction, so $b \notin N$ and $b = x - a \in L \oplus N$. Define $S = \{m \in M: g(m) \in L \oplus N\}$, S is an $R_\Gamma -$ submodule contains K , take $t \in M$ such that $g(t) = b$, then $t \in S$ but $t \notin K$, if $\pi: L \oplus N \to N$ is the projection $R_\Gamma -$homomorphism , then $\pi g: M \to N$ and $(\pi g)(k) = \pi(g(k)) = \pi(f(k)) = f(k)$ for each $k \in K$, thus πg extending of f which is contradiction, therefore $g(M) \subseteq N$.

Corollary 3.11 Every closed $R_\Gamma -$ submodule N of an quasi-injective $R_\Gamma -$ module M is a direct summand of M, moreover , N is quasi-injective.

Proof. Let $I_N: N \to N$ identity map of N. Then by proposition(3.10) there exists $f: M \to N$ such that $fi = I_N$ where i is inclusion mapping, so $\mathrm{Im}(i) \oplus \mathrm{Ker}(f) = M$, hence $N \oplus \mathrm{Ker}(f) = M$. By lemma(1.5) in [1], we have N is quasi-injective.

Corollary 3.12 Let M be $R_\Gamma -$ module. Then M is quasi-injective if and only if $M \oplus M$ is quasi-injective .

Proof. If M is quasi-injective $R_\Gamma -$ module, then by [1, proposition 1.4] M is $M \oplus M -$ injective , by [1, lemma 1.5] $M \oplus M$ is quasi-injective.

The proof of the following propositions follow from proposition(1.3) in [1].

Proposition 3.13 An $R_\Gamma -$ module M is quasi-injective if and only if for each $R_\Gamma -$ submodule B of a cyclic $R_\Gamma -$ submodule A, each $R_\Gamma -$homomorphism $\alpha: B \to M$ can be extending to an $R_\Gamma -$ homomorphism $\beta: A \to M$.

Examples and Remarks 3.14

1. Every simple $R_\Gamma -$ module is quasi-injective.
2. Every injective $R_\Gamma -$ module is quasi-injective, the converse is not true, for example, let $R = \Gamma = Z$ and $M = Z_2$, then M is quasi-injective from(1) but not injective since $1 \neq 2.m.x$ for any $x \in M$, so it is not divisible [1].
3. Let F be a field, $R = \left\{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}: x,y,z \in F\right\}$, and $\Gamma = \left\{\begin{pmatrix} \gamma & \beta \\ 0 & \lambda \end{pmatrix}: \gamma,\beta,\lambda \in F\right\}$. R is a $\Gamma -$ ring with usual multiplication of matrices, consider $A = \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}: a \in F\right\}$, $B = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}: b \in F\right\}$ and $C = \left\{\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}: b,c \in F\right\}$, then $M = A \oplus C$, and $B \cong A$, from corollary(3.9),R is not quasi-injective $R_\Gamma -$ module.
4. Direct sum of two quasi-injective $R_\Gamma -$ modules need not be quasi-injective, for example,let $R = \Gamma = Z$, $M_1 = Z_2$ and $M_2 = Q$ from example(2) and example(2.3) in [1] M_1 and M_2 are quasi-injective. But $M = M_1 \oplus M_2$ is not quasi-injective, since if the $R_\Gamma -$ homomorphism $f: 0 \oplus Z \to M$ by $f(0,n) = (\bar{n},0)$ extended to $R_\Gamma -$ endomorphism g of M, then $g(0,1) = g(2.1.\left(0,\tfrac{1}{2}\right)) = 2.1.g(0,\tfrac{1}{2}) = 2.1.(x,y) = (2.1.x, 2.1.y) = (2x,2y)$, so $(1,0) = g(0,1) = (2x,2y) = (0,2y)$ contradiction.
5. If $R_\Gamma -$ module M contains a copy of R as $R_\Gamma -$module, then M is quasi-injective if and only if M is injective.
6. An $R_\Gamma -$ module M is quasi-injective if and only if for each essential $R_\Gamma -$submodule N of M, each $R_\Gamma -$ homomorphism $f: N \to M$ can be extended to $R_\Gamma -$ endomorphism of M. For each $R_\Gamma -$submodule N of M and each $R_\Gamma -$homomorphism $f: N \to M$, define $g: N \oplus N^c \to M$by $g(n + n') = f(n) + n'$ where $n \in N$ and $n' \in N^c$ since $N \oplus N^c$ is essential by lemma(3.4), then g can be extended to $R_\Gamma -$ endomorphism h of M, clear that h is extending of f.

Lemma 3.15 Let M be a quasi-injective $R_\Gamma -$ module. Then M is injective if there exist an $R_\Gamma -$ epimorphism from M to $E(M)$.

Proof. Let $f: M \to E(M)$ be an $R_\Gamma -$ epimorphism. Then there exists an $R_\Gamma -$ endomorphism $h: E(M) \to E(M)$ such that $f = hi$, since M is quasi-injective, then $h(M) \subseteq M$, hence $E(M) = f(M) = hi(M) = h(M) \subseteq M$, so $M = E(M)$, Thus M is injective $R_\Gamma -$ module.

The annihilator of a left $R_\Gamma -$ module M define by $Ann_R(M) = \{r \in R: r\Gamma M = 0\}$ and the annihilator of $m \in M$ define by $Ann_R(m) = \{r \in R: r\Gamma m = 0\}$ [4]. We denote $\ell_{R_\Gamma}(M)$ and $\ell_{R_\Gamma}(m)$ instead of $Ann_R(M)$ and $Ann_R(m)$.

Definition 3.16 Let M be an $R_\Gamma -$ module, $a \in M$ and $\gamma \in \Gamma$. The left Annihilator of m in R with respect to γ define by $\ell^{\gamma}_{R_\Gamma}(m) = \{r \in R: r\gamma m = 0\}$.

It's clear that $\ell_{R_\Gamma}(M) \subseteq \ell_{R_\Gamma}(m) \subseteq \ell^{\gamma}_{R_\Gamma}(m)$, in fact $\ell_{R_\Gamma}(M) = \bigcap_{m\in M} \ell_{R_\Gamma}(m) = \bigcap_{\substack{m\in M \\ \gamma\in\Gamma}} \ell^{\gamma}_{R_\Gamma}(m)$.

The following proposition gives a characterization of quasi-injective gamma modules.

Proposition 3.17 An $R_\Gamma -$ module M is quasi-injective if and only if for each left ideal L of R, each $R_\Gamma -$ homomorphism $f: L \to M$ with $\ell^{\gamma_\circ}_{R_\Gamma}(a) \subseteq Ker(f)$ for some $a \in M$, f can be extending to an $R_\Gamma -$ homomorphism from R to M.

Proof. Assume M is quasi-injective $R_\Gamma -$ module, L a left ideal of R and f an $R_\Gamma -$ homomorphism from L to M with $\ell^{\gamma_\circ}_{R_\Gamma}(a) \subseteq Ker(f)$ for some $a \in M$. Then $L\gamma_\circ a$ is $R_\Gamma -$ submodule of $\langle a \rangle$, define $\alpha: L\gamma_\circ a \to M$ by $\alpha(r\gamma_\circ a) = f(r)$ for any $r\gamma_\circ a \in L\gamma_\circ a$, if $r\gamma_\circ a = 0$, then $\ell^{\gamma_\circ}_{R_\Gamma}(a) \subseteq Ker(f)$, so $f(r) = 0$, hence α is well defined and easily to show α is $R_\Gamma -$ homomorphism, then by proposition(3.13) α can extends to an $R_\Gamma -$ homomorphism $\beta: \langle a \rangle \to M$, define $g: R \to M$ by $g(r) = \beta(r\gamma_\circ a)$ for each $r \in R$. Since $f(r) = \alpha(r\gamma_\circ a) = \beta(r\gamma_\circ a) = g(r)$ for each $r \in L$, then g extended to f. Conversely, Assume B is any $R_\Gamma -$ submodule of M and $\alpha: B \to M$ is $R_\Gamma -$homomorphism. By Zorn's lemma there exists a maximal element $(B_\circ, \alpha_\circ)$ such that $B \leq B_\circ$ and $\alpha_\circ$ extends of α to $B_\circ$, if $B_\circ = M$ the proof complete, if not there exists $a \in M$ and $a \notin B_\circ$, take $L = \{r \in R: r\gamma_\circ a \in B_\circ\}$, then L is left ideal of R, define $\varphi: L \to M$ by $\varphi(r) = \alpha_\circ(r\gamma_\circ a)$ for each $r \in L$, if $r = 0$, then $r\gamma_\circ a = 0$, so $0 = \alpha_\circ(r\gamma_\circ a) = \varphi(r)$, therefore φ is well define and it is $R_\Gamma -$ homomorphism and for each $r \in \ell^{\gamma_\circ}_{R_\Gamma}(a)$, then $r\gamma_\circ a = 0$, so $0 = \alpha_\circ(r\gamma_\circ a) = \varphi(r)$, so $r \in Ker(\varphi)$, thus $\ell^{\gamma_\circ}_{R_\Gamma}(a) \subseteq Ker(\varphi)$, by hypothesis φ extended to $R_\Gamma -$ homomorphism $\lambda: R \to M$. Define $R_\Gamma -$ submodule C by

$C = B_\circ + R\gamma_\circ a$ and $\beta: C \to M$ by $\beta(b_\circ + r\gamma_\circ a) = \alpha_\circ(b_\circ) + \lambda(r)$ for each $b_\circ \in B_\circ$ and $r \in R$, if $b_\circ + r\gamma_\circ a = 0$, then $b_\circ = -r\gamma_\circ a \in B_\circ$, so $r \in L$ and hence $\lambda(r) = \varphi(r) = \alpha_\circ(r\gamma_\circ a) = -\alpha_\circ(b_\circ)$, thus $\beta(b_\circ + r\gamma_\circ a) = 0$, so β is well defined and it is $R_\Gamma -$ homomorphism , for each $b \in B$, $\alpha(b) = \alpha_\circ(b) = \alpha_\circ(b) + \lambda(0) = \beta(b)$,a contradiction with maximality of $(B_\circ, \alpha_\circ)$, so $B_\circ = M$, thus M is quasi-injective.

Proposition 3.18 If M is an $R_\Gamma -$module ,$E = End_{R_\Gamma}(E(M))$ and $Q = M\Gamma E$, where $M\Gamma E = \{x\gamma f: x \in M, \alpha \in \Gamma$ and $f \in E\}$, then:

1. Q is an quasi-injective $R_\Gamma -$ submodule of E(M) containing M.
2. Q is the intersection of quasi-injective $R_\Gamma -$ submodule of E(M) containing M.
3. $M = Q$ if and only if M is quasi-injective.
4. Q is the smallest quasi-injective $R_\Gamma -$ submodule of E(M) that contains M, furthermore, Q is essential extension of M.

Proof.

1. Since $M = 1_{E(M)}(M) = 1_{E(M)}(1\gamma_\circ M) = M\gamma_\circ 1_{E(M)} \subseteq Q$, this shows that Q contains M . For all $x \in M$, $\alpha \in \Gamma$ and $f \in E$, then $x\gamma f = f(1\gamma x) \in E(M)$, so $Q \subseteq E(M)$, clearly that Q is an $R_\Gamma -$ submodule of E(M). If N is an $R_\Gamma -$ submodule of Q and $f: N \to Q$ is an $R_\Gamma -$ homomorphism, by injectivity of E(M), there exists $\varphi: E(M) \to E(M)$ which extends f. Since $\varphi(x\gamma f) = \varphi(f(1\gamma x)) = \varphi(1\gamma_\circ f(1\gamma x)) = (\varphi\gamma_\circ f)(1\gamma x) = x\gamma(\varphi\gamma_\circ f) \in Q$ for $x \in M$, $\alpha \in \Gamma$ and $f \in E$, therefore $\varphi(Q) \subseteq Q$, so if define $\overline{\varphi} = \varphi_{|Q}$, then $\overline{\varphi}_{|N} = f$ and thus Q is quasi-injective $R_\Gamma -$ module.
2. Let Q' be a quasi-injective $R_\Gamma -$ submodule of E(M) containing M , By proposition(3.7) and part(1) $f(Q') \subseteq Q'$ for $f \in E$, since $M \subseteq Q'$,then $Q = M\Gamma E \subseteq Q'\Gamma E \subseteq E(1\Gamma Q') \subseteq E(Q') \subseteq Q'$ and this shows that Q is the smallest one. Now for any family of a quasi-injective $R_\Gamma -$ submodules $\{Q_\alpha\}_{\alpha\in\Lambda}$ of E(M) each of which contains M , then $Q \subseteq \cap_{\alpha\in\Lambda} Q_\alpha$, but $\cap_{\alpha\in\Lambda} Q_\alpha \subseteq Q$, since $Q \in \{Q_\alpha\}_{\alpha\in\Lambda}$. Thus $Q = \cap_{\alpha\in\Lambda} Q_\alpha$.
3. Follows from(1) and from(2).
4. It's Clear from(2) Q is the smallest quasi-injective $R_\Gamma -$ submodule of E(M) contains M , since M essential in E(M), hence Q is essential in E(M) [1].

Definition 3.19 Let M be an $R_\Gamma -$ module. A quasi-injective hull of M denoted by Q(M) is a quasi-injective $R_\Gamma -$ module containing M such that for any $R_\Gamma -$ monomorphism f from M into a quasi-injective $R_\Gamma -$ module N , extends to an $R_\Gamma -$ monomorphism from Q(M) into N . In fact, $Q(M) = M\Gamma End_{R_\Gamma}(E(M))$.

Lemma 3.20 Every $R_\Gamma -$ module has a quasi-injective hull which is unique up to isomorphism.

Proof. Let M be $R_\Gamma -$module , for each quasi-injective extension N of M and $R_\Gamma -$ monomorphism f from M into N , let $E = End_{R_\Gamma}(E(M))$, $E^* = End_{R_\Gamma}(E(N))$ and $Q = M\Gamma E$. By proposition(3.7), we have $N\Gamma E^* \subseteq N$. Since E(N) is injective $R_\Gamma -$ module , there exists $R_\Gamma -$ homomorphism $g: Q \to E(N)$ such that $gi_M = i_N f$ where $i_M(i_N)$ is the inclusion mapping of M(N) into $Q(E(N))$, if $x \in Ker(g) \cap M$, then $f(x) = g(x) = 0$, so $f(x) = 0$, hence $x \in Ker(f) = 0$, thus g is $R_\Gamma -$ monomorphism, hence g(Q) is quasi-injective and so $E^*\Gamma(g(Q)) \subseteq g(Q)$, take $X = N \cap g(Q)$, then $X\Gamma E^* \subseteq X$, so by proposition(3.7), X is quasi-injective, hence $g^{-1}(X)$ is quasi-injective $R_\Gamma -$ submodule of E(M) contains M , by proposition(3.18) $Q = g^{-1}(X)$, hence $g(Q) = X \subseteq N$. If there exists another quasi-injective hull T of M , then there exists an $R_\Gamma -$ monomorphism $g': T \to Q$ such that $g'f = i$ where i inclusion mapping from M to Q, for each $x\gamma h \in Q$, $g'g(x\gamma h) = g'g(x\gamma h) = g'i_N f(x\gamma h) = h(1\gamma x) = x\gamma h$, so $g'g = I_Q$, hence g' is $R_\Gamma -$ isomorphism.

Definition 3.21 Let M be an $R_\Gamma -$ module and I a left ideal of R. M is called $I -$ bounded if for each left ideal J of R, there exists an element m in M with $\ell_{R_\Gamma}(m) \leq J$ if and only if $I \leq J$.

Every $R_\Gamma -$ module M is $R -$ bounded, since $0 \in M$ and $\ell_{R_\Gamma}(0) = R$ and M is $0 -$ bounded if there exists an element m in M with $\ell_{R_\Gamma}(m) \leq J$ for each ideal J of R.

Remarks 3.22 Let I be a left ideal of a $\Gamma -$ ring R and M is an $I -$ bounded $R_\Gamma -$ module. Then

1. I is the minimal ideal of R with the property $I = \ell_{R_\Gamma}(m)$ fore some $m \in M$. Since $I \leq I$, so by definition(3.21), there exists $m \in M$ such that $\ell_{R_\Gamma}(m) \leq I$. On the other hand, since $\ell_{R_\Gamma}(m) \leq \ell_{R_\Gamma}(m)$ again by definition(3.21)

we have $I \leq \ell_{R_\Gamma}(m)$, so $I = \ell_{R_\Gamma}(m)$. For the minimality, if there exists a left ideal I_1 of R such that $I_1 = \ell_{R_\Gamma}(x)$ fore some $x \in M$, since $\ell_{R_\Gamma}(x) \leq I_1$, by definition(3.21) we have $I \leq I_1$.
2. I is two-sided ideal of R. Since $I = \ell_{R_\Gamma}(m)$, then $I\Gamma M \leq I\Gamma m = 0$. So $(I\Gamma R)\Gamma M = I\Gamma(R\Gamma M) \leq I\Gamma M = 0$, hence $I\Gamma R \leq \ell_{R_\Gamma}(M) \leq \ell_{R_\Gamma}(m) = I$, therefore I is two-sided ideal.
3. Suppose there exists $m_1 \in M$ such that $I \nleq \ell_{R_\Gamma}(m_1)$, then there is no element $x \in M$ such that $\ell_{R_\Gamma}(x) \leq \ell_{R_\Gamma}(m_1)$ which is a contradiction. Thus $I \leq \bigcap_{m\in M} \ell_{R_\Gamma}(m)$ for each $m \in M$.
4. Since $I \leq \ell_{R_\Gamma}(m)$ for each $m \in M$, then$I \leq \bigcap_{m\in M} \ell_{R_\Gamma}(m) = \ell_{R_\Gamma}(M)$, so M is $(R/I)_\Gamma -$ module by the rule $(r + I, \gamma, m) \longmapsto r\gamma m$ for each $r \in R$, $\gamma \in \Gamma$ and $m \in M$ [2].

The following proposition gives a characterization of quasi-injective gamma modules.

Theorem 3.23 Let M be an $I -$ bounded $R_\Gamma -$ module. Then M is quasi-injective if and only if it is injective as an $(R/I)_\Gamma -$ module.

Proof. Assume M is quasi-injective $R_\Gamma -$module. Let K/I is an $R_\Gamma -$ submodule of R/I and $f: K/I \to M$ an $R_\Gamma -$ homomorphism. Define $\alpha: K \to M$ by $\alpha(r) = f(r + I)$ for each $r \in K$, if $r = 0$, then $f(r + I) = 0$, hence α is well-defined and it's easily to show that α is $R_\Gamma -$ homomorphism. Since $I \leq Ker(f)$, then $I = \ell_{R_\Gamma}(M) \leq Ker(\alpha)$. So $\ell_{R_\Gamma}^{\gamma_\circ}(a) \subseteq Ker(\alpha)$, hence by proposition(3.17) α can extends to an $R_\Gamma -$ homomorphism $\beta: R \to M$. Define $g: R/I \to Q$ by $g(r + I) = \beta(r)$ for each $r \in R$. If $r + I = I$, then $r \in I$, so $\beta(r) = \alpha(r) = 0$, thus $g(r + I) = 0$, therefore g is well-defined and for each $r \in K$, $g(r + I) = \beta(r) = \alpha(r) = f(r + I)$, then M is injective as $(R/I)_\Gamma -$ module.

Corollary 3.24 Let M is $0 -$ bounded $R_\Gamma -$ module. Then M is quasi-injective if and only if it is injective.

Examples 3.25
1. If R is simple, then R is $0 -$ bounded. Since $R\Gamma R \neq 0$, then there exists a non-zero element $r \in R$, since $r = 1\gamma_\circ r$, so $1 \notin \ell_{R_\Gamma}^{\gamma_\circ}(r)$, thus $\ell_{R_\Gamma}^{\gamma_\circ}(r) \neq R$, hence $\ell_{R_\Gamma}(r) = 0$. In particular, Z_2 as $(Z_2)_Z -$ module is $0 -$ bounded.
2. The $Z_Z -$ module Z_2 is not $0 -$bounded .Take the ideal $J = 3Z$, since $Z_2 = \{0,1\}$, $\ell_{R_\Gamma}(0) = Z$ and $\ell_{R_\Gamma}(1) = \{n \in Z: n \text{ is even}\}$, so there is not $m \in Z_2$ such that $\ell_{R_\Gamma}(m) \leq J$ but $0 \leq J$. By example(3.14)(2) Z_2 is quasi-injective but not injective, this example show that the condition of $0 -$ bounded in corollary(3.24) cannot be dropped.
3. The $Z_Z -$ module Z is $0 -$ bounded for each ideal J of Z . Since $\ell_{R_\Gamma}(n) = 0 \leq J$ for any nonzero n in Z.
4. Let $R = Z_{12}$, $\Gamma = Z$ and $M = Z_{12}$, take $I_1 = \{0\}$, $I_2 = \{0,6\}$, $I_3 = \{0,4,8\}$, $I_4 = \{0,3,6,9\}$, $I_5 = \{0,2,4,6,8,10\}$, then $\ell_{R_\Gamma}(0) = Z_{12}\ell_{R_\Gamma}(1) = \ell_{R_\Gamma}(5) = \ell_{R_\Gamma}(7) = \ell_{R_\Gamma}(11) = I_1$, $\ell_{R_\Gamma}(2) = I_2$,$\ell_{R_\Gamma}(3) = \ell_{R_\Gamma}(9) = I_3$, $\ell_{R_\Gamma}(4) = \ell_{R_\Gamma}(8) = I_4$, $\ell_{R_\Gamma}(6) = I_5$, hence Z_{12} is $I_1 -$ bounded, $I_2 -$ bounded, $I_3 -$ bounded, $I_4 -$ bounded, $I_5 -$ bounded and Z_{12} – bounded. So Z_{12} is injective as $\left(Z_{12}/I_j\right)_Z -$ module (j=1,2,…,6) by theorem(3.23).

Lemma 3.26 If direct sum of every pair of quasi-injective $R_\Gamma -$ modules is quasi-injective, then every quasi-injective is injective.

Proof. For any ideal I of R and $R_\Gamma -$ homomorphism $f: I \to M$, since $M \oplus E(R)$ is quasi-injective, then there exists an $R_\Gamma -$ endomorphism g of M such that $i_M f = g i_R i_I$ where $i_M(i_R, i_I)$ is the inclusion mapping of $M(R, I)$ into $M\oplus E(R)(M\oplus E(R), R)$. Define $\bar{g}: R \to M$ by $\bar{g} = \pi_1 g i_R$ where π_1 is the projection of $M\oplus E(R)$ into M, then $\bar{g} i_I(n) = \pi_1 g i_R i_I(n) = \pi_1 i_M f(n) = f(n)$ for each $n \in I$, so by proposition(1.7) in [1] M is injective.

Let R be a $\Gamma -$ring, the radical $J(R)$ of R is the set of all elements of R which annihilates all simple $R_\Gamma -$ modules [6]. An element a in $\Gamma -$ring R is called left quasi-regular if there exists a' in R such that $a + a' + a'\gamma a = 0$ for each $\gamma \in \Gamma$, an ideal I of R is left quasi-regular if each its elements is left quasi-regular [10].

Theorem 3.27 [10] Let R be $\Gamma -$ring. Then the radical $J(R)$ of R is left quasi-regular ideal of R contains every left quasi-regular ideal of R.

An element x of a $\Gamma-$ ring R is called regular if there exists $s \in R$ such that $x = x\alpha s\gamma x$ for some $\gamma, \alpha \in \Gamma$ and R is regular if each element of R is regular [8].

Theorem 3.28 Let M be a quasi-injective $R_\Gamma -$ module and $E = End_{R_\Gamma}(M)$, then $J(E) = \{f \in E: Ker(f)$ is essential $R_\Gamma -$ submodule of $M\}$ and $E/J(E)$ is regular $\Gamma -$ ring.

Proof. Let $K = \{f \in E: Ker(f)$ essential $R_\Gamma -$ submodule $M\}$, for each $f, g \in K$, since $Ker(f) \cap Ker(g) \subseteq Ker(f - g)$, then $Ker(f - g)$ is essential $R_\Gamma -$ submodule of M , so $f - g \in K$ and for each $f \in K, \gamma \in \Gamma$ and $h \in E$, if N is non-zero $R_\Gamma -$ submodule of M , since $Ker(f) \leq_e M$, then $h^{-1}(Ker(f)) \leq_e M$ by [1, lemma(3.3)], so there exists $n(\neq 0) \in N \cap h^{-1}(Ker(f))$, hence $h(n) \in Ker(f)$, thus $1\gamma h(n) = h(1\gamma n) \in Ker(f)$, so $(f\gamma h)(n) = f(h(1\gamma n)) = 0$, hence $n \in Ker(f\gamma h)$, therefore $n \in N \cap Ker(f\gamma h) \neq 0$, so $Ker(f\gamma h) \leq_e M$, hence $f\gamma h \in K$, thus $E\Gamma K \subseteq K$, this show that K is an ideal of E. Now for each $f \in K$, define an $R_\Gamma -$homomorphism $h: M \to M$ by $h(x) = 1\gamma f(x)$ for each $x \in M$, since $Ker(f) \subseteq Ker(h)$, then $Ker(h) \leq_e M$ but $Ker(h) \cap Ker(I - h) = 0$ where $I = id(M)$, hence $Ker(I - h) = 0$ and $I - h: M \to Im(I - h)$ is an $R_\Gamma -$isomorphism, so there exists $g: M \to M$ such that $g_{|(I-h)(M)} = (I - h)^{-1}$, hence $g(I - h) = I$, define an $R_\Gamma -$homomorphism $t: M \to M$ by $t(x) = g(x) - f(x) - I(x)$ for each $x \in M$, then $g(x) = t(x) + f(x) + I(x)$, so $I = g - gh = t + f + I - g\gamma f$, hence $t + f - g\gamma f = 0$ for each $\gamma \in \Gamma$, therefore f is quasi-regular by theorem(3.27) $f \in J(E)$, thus $K \subseteq J(E)$. For each $f \in J(E)$, let $K \leq M$ with $K \cap Ker(f) = 0$, then $f' = -f_{|K}: K \to M$ is an $R_\Gamma -$monomorphism , so there exists $g: M \to M$ such that $k = g(f'(k)) = g(1\gamma_\circ f'(k)) = (g\gamma_\circ f')(k)$ for each $k \in K$, hence $(I + g\gamma_\circ f)(K) = 0$, so $K \leq Ker(I + g\gamma_\circ f)$, since $f \in J(E)$ and $J(E)$ is an ideal [10], then $g\gamma_\circ f \in J(E)$, then $g\gamma_\circ f$ is quasi-regular by theorem(3.11) in [10], thus there exists $h \in E$ such that $g\gamma_\circ f + h + h\gamma_\circ g\gamma_\circ f = 0$, that is , $(I + h)\gamma_\circ(I + g\gamma_\circ f) = I$, hence $Ker(I + g\gamma_\circ f) = 0$ but $K \leq Ker(I + g\gamma_\circ f)$, so $K = 0$, therefore $Ker(f) \leq_e M$, thus $J(E) \subseteq K$, so $K = J(E)$.

For each $\bar{f} = f + K \in E/K$, take $B = (Ker(f))^c$ in M , since $Ker(f_{|B}) = 0$, then $f_{|B}$ is an $R_\Gamma -$monomorphism and ${f_{|B}}^{-1}: f(B) \to B$ is $R_\Gamma -$ isomorphism, so $f^{-1}_{|f(B)}$ can be extended to $R_\Gamma -$homomorphism $g: M \to M$ such that $g_{|f(B)} = {f_{|f(B)}}^{-1}$, so $(g\gamma_\circ f)(b) = g(f(1\gamma_\circ b)) = g(f(b)) = b$ for each $b \in B$, hence $g\gamma_\circ f = id(B)$, since $(f\gamma_\circ g\gamma_\circ f - f)(B) = (f\gamma_\circ g)(f(B)) - f(B) = 0$, then $(f\gamma_\circ g\gamma_\circ f - f)(B + Ker(f)) = 0$, so $B \oplus Ker(f) \leq Ker(f\gamma_\circ g\gamma_\circ f - f)$ but $B \oplus Ker(f) \leq_e M$, then $Ker(f\gamma_\circ g\gamma_\circ f - f) \leq_e M$ and $f\gamma_\circ g\gamma_\circ f - f \in K$, so $f\gamma_\circ g\gamma_\circ f + K = f + K$, take $\bar{f} = \bar{f}\gamma_\circ \bar{g}\gamma_\circ \bar{f}$, hence E/K is regular , thus $E/J(E)$ is regular .

References:

1. **M. S. Abbas, S. A. Saadi, E. A. Shallal**, (2016), Injective gamma modules, Annals of Pure and Applied Mathematics, Vol. 12, No. 1, 85-94.
2. **R. Ameri and R.Sadeghi,** (2010), Gamma Modules, Ratio Mathematica, Vol. 20, pp. 127-147.
3. **W. E. Barnes,**(1966),On The Γ −Ring of Nobusawa, Pacific Journal of Mathematics,Vol.18,No.3,pp. 411-422.
4. **M. Dumitru,** (2009), Gamma-Rings: some interpretations used in the study of their radicals, U. P. B. Bull., Vol. 71, No. 3.
5. **L. Fuchs,** (1969), On Quasi-injective modules, Ann. Scuola Norm. Sup. Pisa, 23, 541-546.
6. **S. Kyuno,** (1977),On the semi-simple gamma rings, Tohoku Math. Journ. , 29, 217-225.
7. **S. Kyuno,** (1979),A gamma ring with the right and left unites, Math. Japon. , 24, 191-193.
8. **X. Ma, J. Zhan,**(2010),Some Characterizations of regular semisimple Γ −Ring,Kyungpook Math.J.,50,411-417.
9. **N. Nobusawa ,** (1964), On a Generalization of the ring theory, Osaka Journal Math., Vol. 1,pp 81-89.
10. **A. C. Paul ,Md. SaburUddin,**(2009), On Jacobson radical for $\Gamma -$ Rings, J. Bangladesh Math. Soc.,29,147-160.