Journal Homepage: -www.journalijar.com

INTERNATIONAL JOURNAL OF ADVANCED RESEARCH (IJAR)

Article DOI:10.21474/IJAR01/12751
DOI URL: http://dx.doi.org/10.21474/IJAR01/12751

ISSN NO. 2320-5407

RESEARCH ARTICLE

FINITE ELEMENT DISCRETIZATION OF THE BEAM EQUATION

Hagai Amakobe James
School of Mathematics and Actuarial Science, Maseno University P. O. Box-Private Bag Maseno-Kenya.

……………………………………………………………………………………………………....

Manuscript Info

……………………….

Manuscript History
Received: 16 February 2021
Final Accepted: 19 March 2021
Published: April 2021

Key words:-
Beam, Finite Element Method (Fem), Gerlerkin's Method, Stiffness, Mass, Nodal

Abstract

………………………………………………………………….

A beam is a structural element or member designed to support loads applied at various points along the element. Beams make up a structure which is an assembly of a number of elements. Beams undergo displacement such as deflection and rotations at certain important location of a structure such as centre of a bridge or top of a building. I haveanalysed numerically a two dimensional beam equation with one degree of freedom of the form $u_{tt} + c^2 u_{xxxx} = f(x, t)$ using finite element method. The positive constant c^2 has the meaning of flexural rigidity per linear mass density, $u(x,t)$ the beam deflection and $f(x,t)$ is the external forcing term. This involved discretization of the beam equation employing Galerkin's technique which yields a system of ordinary differential equations.

Copy Right, IJAR, 2021,. All rights reserved.

……………………………………………………………………………………………………....

Introduction:-

The flexture of the uniform elastic length L whose ends are simply supported can be modeled by the equation:

$$\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} = f(x,t) \quad (1)$$

Subject to boundary conditions:

$$u(0,t) = u(L,t) = 0, u_{xx}(0,t) = u_{xx}(L,t) = 0$$

Discretization

Consider the length AB [4, 6, 11], $\Omega = (0, L)$ subdivided into four elements of equal size.

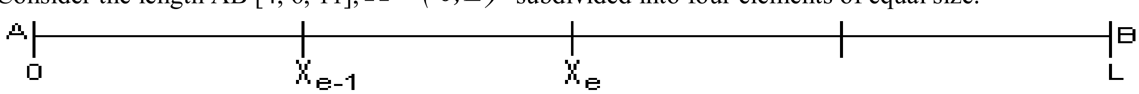

Figure 1:- One dimension domain divided into four elements mesh size=$h_e = x_e - x_{e-1} = \frac{L}{4}$

Corresponding Author:- Hagai Amakobe James
Address:**-** Department of Mathematics and Actuarial Science, Maseno University P. O. Box- Private Bag Maseno-Kenya.

Element approximation function

The function of approximation is given by:

$$\hat{u}(x,t)=\sum_{i=1}^{5}\alpha_i(t)\phi_i(x) \tag{2}$$

where

$\phi_i(x)$ is the shape function or basis function

and $\alpha_i(t)$ the Fourier coefficients then solution is assumed to be in form:

$$\hat{u}(x,t)=\sum_{i=1}^{5}\alpha_i(t)\sin\left(\frac{i\pi x}{L}\right) \tag{3}$$

Formulation over element Ω_e using Galerkin method

$$residual = r(x,t)=\mu\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right)-\mu f(x,t)=0 \tag{4}$$

Multiplying r(x,t) by test function v(x) and varying the integral to weak form gives equation in variation form as in(5) over element Ω_e .In Galerkin method we integrate the product of r(x,t) and v(x) and equate to zero. The number of basis functions will determine the size of the system.

$$\int_{x_{e-1}}^{x_e} r(x,t)v(x)dx=\int_{x_{e-1}}^{x_e} v(x)\left[\mu\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right)-\mu f(x,t)\right]dx=0$$

$$\int_{x_{e-1}}^{x_e}\left[\mu v\frac{\partial^2 u}{\partial t^2}+v\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right)-v\mu f(x,t)\right]dx=0 \tag{5}$$

If assumed approximate solution is;

$$\hat{u}(x,t)=\sum_{i=1}^{5}\phi_i(x)\alpha_i(t) \tag{6}$$

then:

$$\frac{\partial\hat{u}}{\partial t}(x,t)=\sum_{i=1}^{5}\phi_i(x)\dot{\alpha}_i(t) \tag{7}$$

$$\frac{\partial^2\hat{u}}{\partial t^2}=\sum_{i=1}^{5}\phi_i(x)\ddot{\alpha}_i(t) \tag{8}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\hat{u}}{\partial x^2}\right)=\sum_{i=1}^{5}\phi_i^{(IV)}(x)\alpha_i(t) \tag{9}$$

Substituting equations (8) and (9) into (5) we obtain:

$$\int_{x_{e-1}}^{x_e}\left[\mu v\sum_{i=1}^{5}\phi_i(x)\ddot{\alpha}_i(t)+vEI\sum_{i=1}^{5}\phi_i^{(IV)}(x)\alpha_i(t)-v\mu f(x,t)\right]dx=0 \qquad (10)$$

Employing the Galerkin method, then test function

$v(x)=\phi_j(x)$ (11)

where $\phi_j(x)=\dfrac{\sin j\pi x}{L}$ are basis function.

$\phi_i(x)$ and $\phi_j(x)$ is continuously differentiable over the domain $\Omega=[0,L]$ andsatisfy boundary conditions in equation (1). Equation (10) can be divided into three major parts (I, II, III).

$$I=\int_{x_{e-1}}^{x_e}\mu v\sum_{i=1}^{5}v(x)\phi_i(x)\ddot{\alpha}_i(t)dx \quad (12)$$

which gives mass matrix upon integration,

$$II=\int_{x_{e-1}}^{x_e}\sum_{i=1}^{5}v(x)\phi_i^{iv}\alpha_i(t)dx \quad (13)$$

which gives the stiffness matrix upon integration and

$$III=\int_{X_{e-1}}^{X_e}\mu vf(x,t)dx \quad (14)$$

we obtain the nodal force vector upon integration

.

From equation (12) the mass matrix (I) is given by:

$$C_{ij}=\sum_{i=1}^{5}\mu\ \ddot{\alpha}_i(t)\int_{x_{e-1}}^{x_e}\phi_i(x)\phi_j(x)dx \qquad (15)$$

where $\phi_i(x)=sin\dfrac{i\pi x}{L}$,

$$\phi_j(x)=\sin\frac{j\pi x}{L}$$

$$C_{ij}=\sum_{i=1}^{5}\mu\ddot{\alpha}_i(t)\int_{x_{e-1}}^{x_e}\sin\frac{i\pi x}{L}\cdot\sin\frac{j\pi x}{L}dx\ , \qquad (16)$$

From trigonometric identity $SinPxSinQx = \frac{1}{2}\{\cos(P-Q)x - \cos(P+Q)x\}$,then

$$C_{ij} = \sum_{i=1}^{5} \mu \ddot{\alpha}_i \int_{x_{e-1}}^{x_e} \frac{1}{2}\left\{\cos\frac{(i-j)\pi x}{L} - \cos\frac{(i+j)\pi x}{L}\right\} dx \quad (17)$$

upon integration gives:

$$C_{ij} = \sum_{i=1}^{5} \frac{\mu \ddot{\alpha}_i(t)}{2}\left[\frac{L}{(i-j)\pi}\sin(i-j)\pi h_e - \frac{L}{(i+j)\pi}\sin(i+j)\pi h_e\right] (18)$$

For j= 1, 2, 3, 4, 5 ; where $x_e - x_{e-1} = h_e$

The stiffness matrix (II) equation (13) is given by,

$$K_{ij} = \sum_{i=1}^{5} EI \int_{X_{e-1}}^{X_e} v(x) \frac{d^2}{dx^2}\left(\alpha_i(t)\frac{d^2u}{dx^2}\right) dx \quad (19)$$

upon integration by parts we obtain :

$$K_{ij} = \sum_{i=1}^{5} EI \int_{x_{e-1}}^{x_e} \alpha_i(t)\frac{d^2u}{dx^2}\frac{d^2v}{dx^2} dx + \left[\frac{d}{dx}\left\{\alpha_i(t)\frac{d^2u}{dx^2}\right\}v(x) - \alpha_i(t)\frac{d^2u}{dx^2}\frac{dv}{dx}\right]_{x_{e-1}}^{x_e} \quad (20)$$

Inserting boundary conditions from equation (1) into equation (20) we obtain:

$$\frac{d^2u(x_e)}{dx^2} = \frac{d^2u(x_{e-1})}{dx^2} = 0$$

$$K_{ij} = \sum_{i=1}^{5} EI \int_{x_{e-1}}^{x_e} \alpha_i(t)\frac{d^2u}{dx^2}\frac{d^2v}{dx^2} dx \quad (21)$$

where;

$$\frac{d^2v}{dx^2} = \frac{-j^2\pi^2}{L^2}\sin\frac{j\pi x}{L} \quad (22)$$

$$\frac{d^2u}{dx^2} = \frac{-i^2\pi^2}{L^2}\sin\frac{i\pi x}{L} \quad (23)$$

hence,

$$K_{ij} = \sum_{i=1}^{5} EI\frac{j^2 i^2 \pi^4 \alpha_i(t)}{L^4}\int_{X_{e-1}}^{x_e} \sin\frac{j\pi x}{L}\cdot\sin\frac{i\pi x}{L} dx$$

$$K_{ij} = \sum_{i=1}^{5} \frac{EIj^2 i^2 \pi^4 \alpha(t)}{L^4} \int_{x_{e-1}}^{x_e} \frac{1}{2}\left\{\cos\frac{(i-j)\pi x}{L} - \cos\frac{(i+j)\pi x}{L}\right\}dx \quad .(24)$$

Upon integration we obtain:

$$K_{ij} = \sum_{i=1}^{5} EI \frac{i^2 j^2 \pi^3}{2L^3}\alpha_i(t)\left[\frac{1}{i-j}\sin\frac{(i-j)\pi h_e}{L} - \frac{1}{i+j}\sin\frac{(i+j)\pi h_e}{L}\right] \tag{25}$$

where $x_e - x_{e-1} = h_e$ and j= 1,2,3,4,5.
The nodal force (III) in equation (14) given by:

$$F_j = \int_{x_{e-1}}^{x_e} v(x)\mu f(x,t)dx \tag{26}$$

Upon integration of equation (26) we obtain:

$$F_j = -\mu f(x,t)\frac{L}{j\pi}\cos\frac{j\pi h_e}{L} \tag{27}$$

where $h_e = x_e - x_{e-1}$ and j=1,2,3,4,5 , $f(x,t)$ is piecewise external force and assumed constant.

Formation of global stiffness, mass and nodal force matrices

The summation of mass matrix (I) equation (18) is :

$$C_{ij} = \frac{\mu \ddot{\alpha}_1(t)}{2}\left[\frac{L}{(1-j)\pi}\sin\frac{(1-j)\pi h_e}{L} - \frac{L}{(1+j)\pi}\sin\frac{(1+j)\pi h_e}{L}\right]$$

$$+ \frac{\mu \ddot{\alpha}_2(t)}{2}\left[\frac{L}{(2-j)\pi}\sin\frac{(2-j)\pi h_e}{L} - \frac{L}{(2+j)\pi}\sin\frac{(2+j)\pi h_e}{L}\right]$$

$$+ \frac{\mu \ddot{\alpha}_3(t)}{2}\left[\frac{L}{(3-j)\pi}\sin\frac{(3-j)\pi h_e}{L} - \frac{L}{(3+j)\pi}\sin\frac{(3+j)\pi h_e}{L}\right]$$

$$+ \frac{\mu \ddot{\alpha}_4(t)}{2}\left[\frac{L}{(4-j)\pi}\sin\frac{(4-j)\pi h_e}{L} - \frac{L}{(4+j)\pi}\sin\frac{(4+j)\pi h_e}{L}\right]$$

$$+ \frac{\mu \ddot{\alpha}_5(t)}{2}\left[\frac{L}{(5-j)\pi}\sin\frac{(5-j)\pi h_e}{L} - \frac{L}{(5+j)\pi}\sin\frac{(5+j)\pi h_e}{L}\right] \tag{28}$$

wherei= 1,2,3,4,5.

For j=1

$$C_{i1} = \frac{\mu}{2\pi}\left[-\ddot{\alpha}_1(t)\frac{L}{2} + \frac{1.4142L\ddot{\alpha}_2(t)}{3} + \frac{L\ddot{\alpha}_3(t)}{2} + \frac{5.6568L\ddot{\alpha}_4(t)}{15} + \frac{L\ddot{\alpha}_5(t)}{6}\right] \quad (29)$$

For j=2

$$C_{i2} = \frac{\mu}{2\pi}\left[\frac{1.4142L\ddot{\alpha}_1(t)}{3} + \frac{4.246L\ddot{\alpha}_3(t)}{5} + \frac{2\ddot{\alpha}_4(t)L}{3} + \frac{7.071\ddot{\alpha}_5(t)L}{21}\right] \quad (30)$$

For j=3

$$C_{i3} = \frac{\mu}{2\pi}\left[\frac{L\ddot{\alpha}_1(t)}{2} + \frac{4.2426L\ddot{\alpha}_2(t)}{5} + \frac{L\ddot{\alpha}_3(t)}{6} + \frac{5.6568L\ddot{\alpha}_4(t)}{7} + \frac{L\ddot{\alpha}_5(t)}{2}\right] \quad (31)$$

For j=4

$$C_{i4} = \frac{\mu}{2\pi}\left[\frac{5.6568L\ddot{\alpha}_1(t)}{15} + \frac{2\ddot{\alpha}_2(t)L}{3} + \frac{5.6568\ddot{\alpha}_3(t)L}{7} + \frac{5.6568L\ddot{\alpha}_5(t)}{9}\right] \quad (32)$$

For j=5

$$C_{i5} = \frac{\mu}{2\pi}\left[\frac{L\ddot{\alpha}_1(t)}{6} + \frac{7.071L\ddot{\alpha}_2(t)}{21} + \frac{L\ddot{\alpha}_3(t)}{2} + \frac{5.6568\ddot{\alpha}_4(t)L}{9} - \frac{L\ddot{\alpha}_5(t)}{10}\right] \quad (33)$$

The mass in matrix form is:

$$C_{ij} = \frac{\mu L}{2\pi}\begin{bmatrix} -0.5 & 0.4714 & 0.5 & 0.3771 & 0.1667 \\ 0.4714 & 0 & 0.8485 & 0.6667 & 0.3367 \\ 0.5 & 0.8485 & 0.1667 & 0.8081 & 0.5 \\ 0.3771 & 0.6667 & 0.8081 & 0 & 0.6285 \\ 0.1667 & 0.3367 & 0.5 & 0.6285 & -0.1 \end{bmatrix}\begin{bmatrix} \ddot{\alpha}_1 \\ \ddot{\alpha}_2 \\ \ddot{\alpha}_3 \\ \ddot{\alpha}_4 \\ \ddot{\alpha}_5 \end{bmatrix} \quad (34)$$

The summation of stiffness matrix (II) equation (25) gives:

$$K_{ij} = \frac{EI\pi^3 j^2}{2L^3}\left[\frac{1}{1-j}\sin\frac{(1-j)\pi h_e}{L} - \frac{1}{1+j}\sin\frac{(1+j)\pi h_e}{L}\right]\alpha_1(t)$$

$$+ \frac{EI\pi^3 j^2 \cdot 2^2}{2L^3}\left[\frac{1}{2-j}\sin\frac{(2-j)\pi h_e}{L} - \frac{1}{2+j}\sin\frac{(2+j)\pi h_e}{L}\right]\alpha_2(t)$$

$$+ \frac{EI\pi^3 j^2 \cdot 3^2}{2L^3}\left[\frac{1}{3-j}\sin\frac{(3-j)\pi h_e}{L} - \frac{1}{3+j}\sin\frac{(3+j)\pi h_e}{L}\right]\alpha_3(t)$$

$$+ \frac{EI\pi^3 j^2 \cdot 4^2}{2L^3}\left[\frac{1}{4-j}\sin\frac{(4-j)\pi h_e}{L} - \frac{1}{4+j}\sin\frac{(4+j)\pi h_e}{L}\right]\alpha_4(t)$$

$$+ \frac{EI\pi^3 j^2 \cdot 5^2}{2L^3}\left[\frac{1}{5-j}\sin\frac{(5-j)\pi h_e}{L} - \frac{1}{5+j}\sin\frac{(5+j)\pi h_e}{L}\right]\alpha_5(t) \quad (35)$$

For j= 1,2,3,4,5, where i=1,2,3,4,5:

For j=1

$$k_{i1} = \frac{EI\pi^3}{2L^3}\left[\frac{-1\alpha_1(t)}{2} + \frac{5.6568\alpha_2(t)}{3} + 4.5\alpha_3(t) + 6.033\alpha_4(t) + 4.167\alpha_5(t)\right] \quad (36)$$

For j=2

$$k_{i2} = \frac{EI\pi^3}{2L^3}\left[1.885\alpha_1(t) + 30.54\alpha_3(t) + 42.67\alpha_4(t) + 33.67\alpha_5(t)\right] \quad (37)$$

For j=3

$$k_{i3} = \frac{EI\pi^3}{2L^3}\left[4.5\alpha_1(t) + 30.54\alpha_2(t) + 13.5\alpha_3(t) + 116.37\alpha_4(t) + 112.5\alpha_5(t)\right] \quad (38)$$

For j=4

$$k_{i4} = \frac{EI\pi^3}{2L^3}\left[6.033\alpha_1(t) + 42.67\alpha_2(t) + 116.37\alpha_3(t) + 254.41\alpha_5(t)\right] \quad (39)$$

For j=5

$$k_{i5} = \frac{EI\pi^3}{2L^3}\left[4.167\alpha_1(t) + 33.67\alpha_2(t) + 112.5\alpha_3(t) + 251.41\alpha_4(t) - 62.5\alpha_5(t)\right] \quad (40)$$

In matrix form the stiffness matrix is written as;

$$K_{ij} = \frac{EI\pi^3}{2L^3}\begin{bmatrix} -0.5 & 1.885 & 4.5 & 6.033 & 4.167 \\ 1.885 & 0 & 30.54 & 42.67 & 33.67 \\ 4.5 & 30.54 & 13.5 & 116.37 & 112.5 \\ 6.033 & 42.67 & 116.37 & 0 & 251.41 \\ 4.167 & 33.67 & 112.5 & 251.41 & -62.5 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} \quad (41)$$

The nodal force vector (III) equation (27) is given by:

$$F_j = -\frac{\mu L f(x,t)}{j\pi}\cos\frac{j\pi h_e}{L} \quad (42)$$

for j=1,2,3,4,5.

Substituting the values j=1,2,3,4,5 gives:

$$F_1 = -\frac{\mu L f(x,t).0.7071}{\pi} \quad (43)$$

$$F_2 = 0 \quad (44)$$

$$F_3 = \frac{\mu f(x,t).0.7071}{3\pi} \quad (45)$$

$$F_4 = \frac{\mu L f(x,t)}{4\pi} \quad (46)$$

$$F_5 = \frac{\mu L f(x,t).0.7071}{5\pi} \quad (47)$$

Then the nodal force vector is:

$$F = \frac{\mu L}{\pi}\begin{bmatrix} -0.7071.f(x,t) \\ 0 \\ 0.2357 f(x,t) \\ 0.25 f(x,t) \\ 0.14142 f(x,t) \end{bmatrix} \quad (48)$$

The equilibrium equation is given by:

$$\frac{\mu L}{2\pi}\begin{bmatrix} -0.5 & 0.4714 & 0.5 & 0.3771 & 0.1667 \\ 0.4714 & 0 & 0.8485 & 0.6667 & 0.3367 \\ 0.5 & 0.8485 & 0.1667 & 0.8081 & 0.5 \\ 0.3771 & 0.6667 & 0.8081 & 0 & 0.6285 \\ 0.1667 & 0.3367 & 0.5 & 0.6285 & -0.1 \end{bmatrix}\begin{bmatrix} \ddot{\alpha}_1 \\ \ddot{\alpha}_2 \\ \ddot{\alpha}_3 \\ \ddot{\alpha}_4 \\ \ddot{\alpha}_5 \end{bmatrix}+$$

$$\frac{EI\pi^3}{2L^3}\begin{bmatrix} -0.5 & 1.885 & 4.5 & 6.033 & 4.167 \\ 1.885 & 0 & 30.54 & 42.67 & 33.67 \\ 4.5 & 30.54 & 13.5 & 116.37 & 112.5 \\ 6.033 & 42.67 & 116.37 & 0 & 251.41 \\ 4.167 & 33.67 & 112.5 & 251.41 & -62.5 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix}=\frac{\mu L}{\pi}\begin{bmatrix} -0.7071.f(x,t) \\ 0 \\ 0.2357f(x,t) \\ 0.25f(x,t) \\ 0.14142f(x,t) \end{bmatrix} \quad (49)$$

Conclusion:-

Exact or / and approximate solution of equation (49) can be obtain by Laplace transform method or numerical method such as finite difference method.

References:-

1. C. S.Krishnanoorthy, (1991) Finite Element AnalysisTata McGraw-Hill publishing CO.
2. Euler-Bernoulli beam equation, April 20, (2009)http://en.wikipedia.org/wiki/Euler-Bernoulli beam equation. Pp 1-20
3. Fair-Weather G., (1978) Finite Element Gerlerkin Methods for Differential Equations MarcelDekker, New York.
4. Majid Mirzael, May 9,(2008) Finite Element Methods 1 lecture notes, http://www.modares.ac.ir/eng/mmirzaei/finite.htm . Pp. 1-40.
5. O.C. Zienkiewicz,(1977) The Finite Element Method, 3rd edition expanded and revised Tata McGraw-Hill publishing Co.Pp 1-200.
6. S.S. Sastry, (1988) Introduction Methods of Numerical Analysis 3rd edition Prentice Hall of India Private Ltd Pp. 325-360 .
7. W.T.Thomson, (1988) Theory of Vibration with Applications 3rd edition CBS Publisher and distributors Pp. 233-240, Pp100-110.
8. W.J.Palm III (1998) Introduction to Matlab for Engineers WCB McGraw-Hill publishing Co.