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RESEARCH ARTICLE

COMPUTING THE MATRIX POWERS OF MATRIX

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Abstract

In this paper, considering fractional matrix power definition, we define matrix powers of matrixusing matrix exponential and matrix logarithm. Finally present a guide for computingthe matrix powers of matrix.

Introduction:-

Matrices are broadly used in computer science, engineering, statistics and economic theory. Theory of matrices isan important area in linear algebra.General linear groups $GL_n(\Bbbk)$, where$\Bbbk = \mathbb{R}$, the real numbers, or $\Bbbk = \mathbb{C}$, the complexnumbers, are considered as both groups and topological spaces.Matrix analysis are used in many areas specially matrix exponential and logarithm. Among the matrix functions matrix exponential is a very useful subclass of functions ofmatrices that has been studied widely. The computation of matrixfunctions has been one of the veryinterestingwork in $GL_n(\Bbbk)$. Among thematrix functions one of the most interesting is the matrix exponential [3]. The principalmatrix power A^α for a matrix $A \in \mathbb{C}^{n\times n}$ and a real number$\alpha \in \mathbb{R}$ is defined by $A^\alpha = \exp(\alpha\, log(A))$ [1] which is a generalization of z^α, where z is a non-zero complex number and α is a complex constant [4].

In this paper, first presentmatrix exponential and matrixlogarithm. Finally, defined matrix power A^B for matrices$A \in N_{M_n(\Bbbk)}(I,1)andB \in GL_n(\Bbbk)$, where$\Bbbk = \mathbb{R}$, the real numbers, or $\Bbbk = \mathbb{C}$, the complexnumbers.

Matrix exponentialand matrix logarithmfor more detail in [2],[3]:-

Let$A \in M_n(\Bbbk)$. The matrix valued series

$$\mathrm{Exp}(A) = \sum_{n\geq 0} \frac{1}{n!} A^n \ (1),$$

and

$$\mathrm{Log}\,(A) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} A^n \qquad (2),$$

have radii of convergence (r. o. c) ∞ and 1 respectively.

Using these series we define the exponential function$\exp: M_n(\Bbbk) \longrightarrow GL_n(\Bbbk)$, by $\exp(A) = \mathrm{Exp}\,(A)$ and $log: N_{M_n(\Bbbk)}(I,1) \longrightarrow M_n(\Bbbk)$, by $\log(A) = Log\,(A - I)$, where $N_{M_n(\Bbbk)}(I,1) = \{A \in M_n(\Bbbk) | \|A - I\| < 1\}$.

i.e. for $A \in M_n(\Bbbk)$,

$$\exp(A) = \sum_{n\geq 0} \frac{1}{n!} A^n \ (3),$$

andfor$\|A - I\| < 1$,

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$$\log(A) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n}(A-I)^n \quad (4).$$

The functions exp and log satisfy the following two properties:

(i) If $\|A-I\| < 1$, then $\exp(\log(A)) = A$ and;

(ii) If $\|\exp B - I\| < 1$,then $\log(\exp(B)) = B$.

Definition of matrix power A^B:-

The matrix powers of matrix is interested can be defined in following way. Inmathematics, the matrix powers of matrix analogous to complex exponents [4].We are now in a position to come up with a meaningful definition of what is meantby a matrix raised to a matrix power.

Let $A \in N_{M_n(\mathbb{k})}(I,1)andB \in GL_n(\mathbb{k})$, where$\mathbb{k} = \mathbb{R}$, the real numbers, or $\mathbb{k} = \mathbb{C}$, the complexnumbers. Then the matrix power of A to B is denoted by A^B and is defined by$A^B = \exp(B\log(A))$.

Computation of matrix exponential and logarithm [2], [6]:-

1. If $D = diag(d_1, d_2, \dots, d_n)$ is a diagonal matrix, thenexp$(D) = diag(e^{d_1}, e^{d_2}, \dots, e^{d_n})$;
2. if A is diagonalizable, i.e.,$P^{-1}AP = D = diag(d_1, d_2, \dots, d_n)$ for some invertible$n \times n$ matrix P, thenexp$(A) = P\exp(D)P^{-1}$;
3. Every $n \times n$ invertible complex matrix, A, has a logarithm, X. To fnd sucha logarithm, we can proceed as follows:
   (i) Compute a Jordan form, $A = PJP^{-1}$, for A and let m be the number of Jordan blocksin J.
   (ii) For every Jordan block, $J_{r_k}(\alpha_k)$, of J, write $J_{r_k}(\alpha_k) = \alpha_k I\,(I + N_k)$, where N_k is nilpotent. Furthermore,$N_k = {\alpha_k}^{-1}H$, where H is the nilpotent matrix of index of nilpotency, r_k, given by
   $$H = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$
   (iii) If $\alpha_k = \rho_k e^{i\theta_k}$, with $\rho_k > 0$,
   let$S_k = \begin{pmatrix} \log\rho_k + i\theta_k & 0 & \cdots & 0 \\ 0 & \log\rho_k + i\theta_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \log\rho_k + i\theta_k \end{pmatrix}.$
   (iv) For everyN_k, let$M_k = N_k - \frac{{N_k}^2}{2} + \frac{{N_k}^3}{3} + \dots + (-1)^{r_k}\frac{{N^{r}}_k{}^{-1}}{r_k - 1}$, where r_k is the index of nilpotency of N_k. We have $I + N_k = e^{M_k}$.
   (v) If$Y_k = S_k + M_k$and $Y$is the block diagonal matrix $diag(Y_1, Y_2, \dots, Y_m)$, thenlog$(A) = PYP^{-1}$.

A Guide for computiong $A^B$by example:-

Let$A = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & 2 & 1 \end{pmatrix}$. Then eigen values of A are $3, 1+2i, 1-2i$. Hence, the Jordan matrix $J$with three blocks $J(3)$,$J(1+2i)$,$J(1-2i)$ is

$J = \begin{pmatrix} J(3) & \dots & 0 \\ \vdots & J(1+2i) & \vdots \\ 0 & \dots & J(1-2i) \end{pmatrix}$, where$J(3) = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$, $J(1+2i) = \begin{pmatrix} 1+2i & 1 \\ 0 & 1+2i \end{pmatrix}andJ(1-2i) = \begin{pmatrix} 1-2i & 1 \\ 0 & 1-2i \end{pmatrix}$. Next we compute the nilpotent matrix $N_k$and $S_k$for each α_k.

When $\alpha = 3, 1+2i, 1-2i$ we get, $N_2 = \frac{1}{3}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $N_2 = \frac{1-2i}{5}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $N_2 = \frac{1+2i}{5}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Hence $M_2 = N_2$.

When $\alpha = 3$: $S_2 = \begin{pmatrix} \log 3 & 0 \\ 0 & \log 3 \end{pmatrix}$.

When $\alpha = 1+2i$: $\rho = \sqrt{5}$ and$\theta = cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$, and hence

$$S_2 = \begin{pmatrix} \log\sqrt{5} + i\,cos^{-1}\left(\frac{1}{\sqrt{5}}\right) & 0 \\ 0 & \log\sqrt{5} + i\,cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \end{pmatrix}.$$

When $\alpha = 1 + 2i$: $\rho = \sqrt{5}$ and$\theta = 2\pi - cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$, hence

$$S_2 = \begin{pmatrix} \log\sqrt{5} + i\left(2\pi - cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\right) & 0 \\ 0 & \log\sqrt{5} + i\left(2\pi - cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\right) \end{pmatrix}.$$

Thus, $Y_1 = \begin{pmatrix} \log 3 & 0 \\ 0 & \log 3 \end{pmatrix}$, $Y_2 = \begin{pmatrix} \log\sqrt{5} + i\,cos^{-1}\left(\frac{1}{\sqrt{5}}\right) & \frac{1-2i}{5} \\ 0 & \log\sqrt{5} + i\,cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \end{pmatrix}$ and

$$Y_2 = \begin{pmatrix} \log\sqrt{5} + i\,cos^{-1}\left(\frac{1}{\sqrt{5}}\right) & \frac{1+2i}{5} \\ 0 & \log\sqrt{5} + i\,cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \end{pmatrix}.$$

$Y = \begin{pmatrix} Y_1 & \dots & 0 \\ \vdots & Y_2 & \vdots \\ 0 & \dots & Y_3 \end{pmatrix}$, and computing coresponding metrix P we can obtain $\log(A)$ by computing PYP^{-1}. Thus, A^B can be computed using$\exp(BPYP^{-1})$.

Conclusion:-

In this work, we defined the matrix powers of matrix for $A \in N_{M_n(\mathbb{k})}(I, 1)$ and$B \in GL_n(\mathbb{k})$, by$A^B = \exp(B\log(A))$,where$\mathbb{k} = \mathbb{R}$, the real numbers, or $\mathbb{k} = \mathbb{C}$, the complexnumbers. UsingJordan block$J_r(\alpha)$) with $\alpha \neq 0$,we can compute$\log(A) = PYP^{-1}$,and henceA^B can be computed using $\exp(BPYP^{-1})$.

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