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RESEARCH ARTICLE

R-ANNIHILATOR-SUPPLEMENTED SUBMODULES.

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Abstract

Let R be associative ring with identity and let M be unitary left R-module. Let U and V be submodules of an R-module M. We say that V is R-annihilator-supplement of U in M if M=U+V and whenever Y≤V and M=U+Y, then ann Y=0.Let M be an R-module. We say that M is R-annihilator -supplemented module if every proper submodule of M has R-a-supplement.Now let U and V be submodules of an R-module M. We say that V is R-annihilator-weak supplement of U in M if M=U+V and $U \cap V \ll^{a} M$.Let M be an R-module. We say that M is R-annihilator weakly supplemented module if every submodule of M has R-a-weak supplement in M. The sum A_M of all such submodules of M.When M is finitly generated and faithful , we needed to A_M in this paper that are relevant to work, when A_M is R-annihilator- small submodules. The main purpose of this work is to develop the properties of R-a-supplemented modules and R-a-weak supplemented modules.

Introduction:-

Throughout this paper all rings are associative ring with identity and modules are unitary left modules.

In [1],[2],[3] R. Wisbauer, C. Lomp and C. Nebiyev defined a supplement and weakly supplement as follows : Let U be a submodule of an R- module M. Recall that a submodule V of M is called a supplement of U in M. If V is a minimal element in the set of submodules L of M with U+L=M. Let M be an R-module, M is called a supplemented module if every submodule of M has a supplement in M. Let U, V be submodules of an R-module M. If M=U+V and $U \cap V \ll M$, then V is called a weak supplement of U in M. Let M be an R-module if every submodule of M has a weak supplement in M, then M is called a weakly supplemented module.

These observation lead us to introduce the following concept: Let V and U be submodules of an R-module M. We say that V is R-annihilator-supplement(R-a-supplemented) of U in M if M=U+V and whenever Y≤V and M=U+Y, then ann Y=0, where $\operatorname{ann}(Y)\ \}=\{r \in R :\ r.Y = 0\}$.

Now let U and V be submodules of an R-module M. We say that V is R-annihilator-weak supplement (R-a-weak supplemented) of U in M if M=U+V and $U \cap V \ll^{a} M$.

In fact, the set K_M of all elements k such that Rk is semisubmodule and annihilator-small .And contains both the Jacobson radical and the singular submodule when M is finitely generated and faithful .

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In [4],[5]. Nicholson, W.K and Amouzegar Kalati,T defined a submodule A_M generated by K_M is a submodule of M analogue of the Jacobson radical that contains every R-annihilator –small submodules .

In section two; we introduce the concept of R-a-supplement submodule with some examples and basic properties. We show that: Let M and M' be R-modules and let f: M→ M' be an epimorphism if M' is R-a-supplemented module, then M is R-a-supplmented module. See(3-2-11).

In section three; we introduce the concept of R-a-weak supplement submodules. We proof that if M is R-a-weakly supplemented module and L is a submodule of M with $L\cap A_M=0$, then L is semisimple.

Characterizations:-
We abbreviate the Jacobson radical as Rad(M) and the singular submodule as Z(M) for any R- module M . The notations $N\leq^e M$ and $N\ll^a M$ mean respectively that a submodule N of M is essential and annihilator small in the module M. See[6], [7].

R- annihilator- supplemented submodule:-
In this section we introduce the concept of R- annihilator supplement (R-a-supplement) submodule, and we illustrate it by examples. We also give some basic properties of this class of submodules.

We start by a definition.

Definition (3.2.1):- Let V and U be submodules of an R-module M. We say that V is R-annihilator-supplement (R-a-supplement) of U in M if M=U+V and whenever $Y\leq V$ and M=U+Y, then ann Y=0.Let M be an R-module. We say that M is R-annihilator -supplemented(R-a-supplemented) module if every proper submodule of M has R-a-supplement.Let R be a commutative ring and let I be an ideal of R. We say that R is R-a-supplemented ring of R is R-a-supplemented as an R-module.

The following proposition gives a characterization of R-a-supplement submodule.

Proposition (3.2.2): Let U and V he submodules of an R-module M. Then V isR-a-supplement of U if and only if M=U+V and $U\cap V \ll^a V$.

Proof: →) Let V be R-a-supplement of U. To show that $U\cap V$ is R-a-small in V, let V= $(U\cap V)$ +Y. Now M= U+V= U+ $(U\cap V)$ +Y= U+Y. But $Y\leq V$, therefore annY=0 and $U\cap V\ll^a V$.
←) Let M=U+V and $U\cap V\ll^a V$. We want to show that V is R-a-supplement of U. Let $Y\leq V$ such that M=U+Y. By Modular law, V= $(U\cap V)$+Y. But $U\cap V\ll^a V$, therefore annY=0. Thus V is R-a-supplement of M.

Examples (3.2.3):-

1- R-a-supplement submodule need not be supplement submodule for example consider Z as Z-module. For every proper submodule nZ of Z. Z=nZ+Z and $nZ\cap Z$= $nZ\ll^a Z$. Hence Z is Z-a-supplement of nZ. Thus every proper submodule of Z has Z-a-supplement. But it is known that every non trivial submodule of Z has no supplement in Z. Where Z is indecomposable and {0} is the only small submodule of Z. One can easily show that a non trivial submodule of Z has more than one Z-a-supplement in Z.

2- A supplement submodule need not be R-a-supplement submodule. For example, consider Z_4 as Z- module. Z_4 is a supplement of $\{\bar{0},\bar{2}\}$. But Z_4 is not Z-a-supplement of $\{\bar{0},\bar{2}\}$,where $\{\bar{0},\bar{2}\}\cap Z_4=\{\bar{0},\bar{2}\}$is not Z-a-small in Z_4, because $Z_4=\{\bar{0},\bar{2}\}+Z_4$ and ann $Z_4=\{n\in Z\ ;\ n.\,Z_4=0\}=4Z\neq 0$
Now, we give some basic properties of R-a- supplement submodules.

Remark(3.2.4): Let M be an R-module. Then every R-a-small submodule of M has R-a-supplement in M.

Proof: Let U be R-a-small submodule of M. Then M=U+M and $U\cap M$=U is R-a-small submodule of M. Thus M is R-a-supplement of U in M.Let M be an R-module it is known that every direct summand of M has a supplement in M. But this is not true for R-a-supplement as the following examples shows:

Example(3.2.5): Consider Z_6 as Z-module.

Let U=$\{\bar{0},\bar{2},\bar{4}\}$ and V=$\{\bar{0},\bar{3}\}$, Z_6= U⊕V. U and V are supplement of each others. But each of U and V has no Z-a-supplement in Z_6, where ann Z_6={n∈Z ; n. Z_6=0}=6Z≠0. Hence Z_6 has no Z-a-small submodule, by prop (3-2-2). Thus every submodule of Z_6 has no Z-a-supplement in Z_6.

Remark(3.2.6): Let R be an integral domain and Let M be a torsion free R-module. Then every proper submodule of M has R-a-supplement in M.

Proof: Let U be a submodule of M. Then M= U+M and U∩M=U is R-a-small submodule of M, by Remark **(3-2-4)**. Thus M is R-a-supplement for U.

We prove the following proposition.

Proposition (3.2.7): Let U and V be two submodules of an R-module M such that V is R-a-supplement of U. If M=W+V, where W is a submodule of U, then V is R-a-supplement of W.

Proof: Since V is R-a-supplement of U, then M=U+V and $U\cap V\ll^a V$. Since$W\leq U$. Then $W\cap V \ll^a V$, by prop (2.4)in [8]. Thus V is R-a- supplement of W in M.

Proposition(3.2.8):
Let M be a finitely generated R-module and let U and V be two submodules of M such that V is R-a-supplement of U in M. Then there exists a finitely generated R-a-supplement V' of U such that $V' \leq V$.

Proof: Let $M=Rx_1+Rx_2+\cdots+Rx_n$, where $x_i \in M$, ∀ i=1,2,…,n. Since M=U+V, then $x_i = u_i + v_i$, where $u_i \in U$, $v_i \in V$, ∀ i=1,2,…,n. Now let $V'=Rv_1+Rv_2+\cdots+Rv_n$. Clearly that $M=U+V'$. Since V is R-a-supplement of U and $V' \leq V$, then annV'=0. One can easily show that V' is R-a-supplement of U.

The following two proposition gives some properties of R-a-supplement submodule.

Proposition(3.2.9):
Let U, V and K be submodules of an R-module M such that V is R-a-supplement of U. If $K\ll^a M$ and

ann $U \leq^e R$, then V is R-a-supplement U+K.

Proof: Since M=U+V, then M=U+K+V. Now let X be a submodule of V such that M=U+K+X. We want to show that ann X=0. Since K is R-a-small in M, then 0=ann(U+X)=annU∩annX. But ann$U\leq^e R$, therefore annX=0. Thus V is R-a-supplement of U+K.

Proposition(3.2.10):
Let M be a faithful R-module and N, K be submodules of M such that N≤K. If K is R-a-supplement submodule in M and ann$N\leq^e R$, then N is R-a-small in K.

Proof: Let K=N+L, where L is a submodule of K. since K is a-supplemented submodule in M , then there exists a submodule K' of M such that $M=K+K'$and $K\cap K' \ll^a K$ and annK=0. Now 0=annK=ann(N+L)=ann(N)∩ann(L). But ann$N\leq^e R$, then ann(L)=0. Thus N is R-a-small submodule of K.

Proposition (3.2.11):
Let M and M' be R-modules and let f: M→ M' be an epimorphism if M' is R-a-supplemented module, then M is R-a-supplmented module.

Proof: Let K be a submodule of M, then f(K) submodule of M'. Since M' isR-a-supplemented module. Then there exists a submodule L of M' such that M'=f(K)+L and f(K)∩L is R-a-small in L. Now $M=f^{-1}(M')=f^{-1}(f(K)+L)=f^{-1}(f(K))+f^{-1}(L)=K+\ker f+f^{-1}(L)=K+f^{-1}(L)$.Claim that $K\cap f^{-1}(L)$ is R-a-small submodule of $f^{-1}(L)$. Since $f(K)\cap L\ll^a L$, then by prop (2-5)in[8], $f^{-1}(f(K)\cap L)\ll^a f^{-1}(L)$.

But $f^{-1}(f(K)\cap L)=f^{-1}(f(K))\cap f^{-1}(L)=(K+\mathrm{Kerf})\cap f^{-1}(L)$
$=\mathrm{kerf}+(K\cap f^{-1}(L))$, by Modular Law . So kerf $+(K\cap f^{-1}(L)) \ll^a f^{-1}(L)$.
By Prop (2-4)in[8], $K\cap f^{-1}(L) \ll^a f^{-1}(L)$. So $f^{-1}(L)$ is R-a-supplement of K in M. Thus M is R-a- supplemented module.

We end this section by the following proposition.

Proposition (3.2.12):

Let M be a finitely generated faithful multiplication module over a commutative ring R and let I be an ideal of R. if IM has R-a-supplement in M, then I has R-a-supplement in R.

Proof: Let I be an ideal of R such that IM has R-a-supplement in M. Then there exists a submodule N of M such that M=IM+N and $IM\cap N \ll_a N$. Since M is a multiplication module, then N=JM, for some ideal J of M. Now M=RM=IM+JM=(I+J)M. But M is finitely generated faithful multiplication module, so M is a cancellation module, by [10] and hence R=I+J. Now $IM\cap N=IM\cap JM=(I\cap J)M\ll_a JM$. To show that $I\cap J\ll_a J$. Let $J=(I\cap J)+K$, where K is an ideal of R. Then $JM=((I\cap J)+K)M=(I\cap J)M+KM$. Therefore ann KM=0. But ann K$\leq$ ann KM. Thus ann K=0 and $I\cap J\ll_a J$. Thus J is R-a- supplement of I.

R-annihilator-weakly supplemented modules:-

In this section, we introduce the definition of R-annihilator-weakly supplemented modules. And we give some basic properties of this module.

We start by the following definition.

Definition (3-3-1):

Let U and V be submodules of an R-module M. We say that V is R-annihilator-weak supplement (R-a-weak supplement) of U in M if M=U+V and $U\cap V\ll^a M$.

Let M be an R-module. We say that M is R-annihilator weakly (R-a-weakly) supplemented module if every submodule of M has R-a-weak supplement in M.

Remarks and examples(3.3.2):-

1. R-a-weak supplement submodule need not be weak supplement submodule. For example, consider Z as Z module. Cleary that M=2Z+3Z and $2Z\cap 3Z=6Z\ll^a Z$, see remark (3-2-4). Thus 3Z is R-a-weak supplement of 2Z. But {0} is the only small submodule of Z and hence 3Z is not weak supplement of 2Z.
2. Every R-a-supplemented module is R-a-weakly supplemented module.To show that, let M be
3. an R-a-supplemented module and let U be a proper submodule of M, then there exists a submodule V of M, such that M=U+V and $U\cap V \ll^a V$. By prop (2-3)in[8], $U\cap V \ll^a M$. Hence V is R-a-weak supplement of U. Clearly that M=M+0 and $M\cap\{0\}=0\ll^a M$. So {0}is R-a-weak supplement of M. Thus M is R-a-weak-supplemented module.
4. R-a-weak supplement submodule need not he R-a-supplement need not be R-a-supplement submodule. For example, let M be a faithful R-module. Then M=M+0 and $M\cap\{0\}=0\ll^a M$. Thus {0} is R-a-weak supplement of M. Now $M\cap\{0\}=0$ is not R-a-small in 0, when {0=0+0 ann0=R$\neq$0}.
5. Let M and R-module of X and Y be submodules of M if X is R-a-weak supplement of Y, then Y is R-a-weak supplement of X, where M=X+Y and $X\cap Y \ll^a M$
6. A semisimple R-module need not be R-a-weak supplemented module. For example, consider Z_6 as
7. Z-module, ann $Z_6=\{n \in Z\ ;\ n.\, Z_6=0\}=6Z\neq 0$. So Z_6 is not faithful and hence has no R-a-small submodule, by remark(3-2-4). Thus every submodule of Z_6 has no R-a-weak supplement in Z_6.

The following proposition gives a condition under which a factor R-a-weak supplement module is R-a-weak supplement.

Proposition(3.3.3):-

Let N, K and W be submodules of an R-module M such that W$\leq$N. If K is R-a-weak supplement of N and M=W+K, there K is R-a-weak supplement of W in M.

Proof: Since K is R-a-weak supplement of N, then M=N+K and N∩K $\ll^a$ M. Now M=W+K and W∩K≤ N∩K$\ll^a$M. Hence W∩K$\ll^a$M by prop (2-4)in[8]. Thus K is R-a-weak supplement of W is M.

Now we give some basic properties of R-a-weak supplement submodule.

Proposition(3.3.4):

Let N and K be submodules of a finitely generated R-module M. if K is R-a-weak supplement of N, then k contains a finitely generated R-a-weak supplement of N.

Proof:- Let $M=Rx_1+Rx_2+\cdots+Rx_n$, $x_i \in M$, for some $x_i \in M$, $\forall$ i=1,2,…,n. Since M=N+K, then $x_i=a_i+b_i$, where $a_i \in N$, $b_i \in K$, $\forall$ i=1,2,…, n. Now let $K'=Rb_1+Rb_2+\cdots+Rb_n$, $M=N+K'$. Clearly that $K' \leq K$. But $N\cap K'=N\cap K\ll^a M$, therefore $N\cap K'\ll^a M$, by prop (2-4)in[8].Thus K' isa finitely generated R-a-weak supplement of N.By the same way of the proof of proposition (3.2.12), we can proof the following.

Proposition(3.3.5):-

Let M be a finitely generated faithful multiplication module over a commutative ring R and let I be an ideal of R. If IM has R-a-weak supplement in M, then I has R-a-weak supplement in R.

The following proposition show that L is semisimple if $L\cap A_M=0$.

Proposition(3.3.6):-

Let M be R-a-weakly supplemented module and let L be a submodule of M with $L\cap A_M=0$, then L is semisimple.

Proof: Let M he R-a-weak supplemented module and let L be a submodule of M such that $L\cap A_M=0$. To show that L is semisimple, let N be a submodule of L. Then there exist a submodule K of M such that M=N+K and $N\cap K\ll^a M$. Now L=L∩M=L∩(N+K). By modular Law L=N+(L∩K). Since $N\cap K\leq A_M$, then $L\cap N\cap K\leq L\cap A_M=0$. So L=N⊕ (L∩K). Thus L is a semisimple module.

Proposition(3.3.7):-

Let M he a finitely generated and faithful R-module such that $A_M \ll^a M$. Let M1 and K be submodule of M such that M_1 is R-a-weakly supplemented module and M_1+K has R-a-weakly supplement in M, then K has R-a-weakly supplement in M.

Proof:- Since M_1+K has R-a-weakly supplement in M_1, then there exists a submodule N of M such that $M=M_1+K+N$ and $(M_1+K)\cap N \ll^a M$. Now

$(K+N)\cap M_1 \leq M_1$ and M_1 is R-a-weakly supplemented module, then there exists a submodule L of M_1 such that $M_1=((K+N)\cap M_1)+L$ and $((K+N)\cap M_1)\cap L \ll^a M$. Then $(K+N)\cap L\ll^a M$.

Now $M= M_1 + K + N =((K+N)\cap M_1) + L+K+N=K+N+L$.

$K\cap(N+L) \leq L\cap(K+N) + N\cap(K+L)$, by [9, lemma(3-2-3), CH3].

Since by prop(3-13)in[8], $(K+L)\cap N \leq (M_1+K)\cap N \ll^a M$, and hence $L\cap(K+N) + N\cap(L+K) \ll^a M$. By remark(2-4)in [8], so $K\cap(N+L) \ll^a M$.Thus N+L is R-a- weak supplement of K in M.

We end this section by the following proposition.

Proposition(3.3.8):

Let M be a finitely generated faithful R-modems such that $A_M \ll^a M$.

If $M=M_1+M_2$, with M1 and $M_2$are R-a-weakly supplemented Moduls, then M is also R-a-weakly supplemented module.

Proof: Assume that M1 and M2 are R-a-weakly supplemented modules of $M=M_1+M_2$. Let N be a submodule of M. Then $M=N+M_1+M_2=(M_1+(N+M_2))+0$. Clearly that M1 + (N+ M) has 0 as an R-a-weakly supplemented in M. Then by prop (3.3.7) M_2+N has R-a-weakly supplemented in M. Since M2 is R-a-weakly supplemented module, then by prop (3.3.7) N has R-a-weakly supplemented in M. Thus M is R-a-weakly supplemented in module.

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