Journal Homepage: -www.journalijar.com

INTERNATIONAL JOURNAL OF ADVANCED RESEARCH (IJAR)

Article DOI:10.21474/IJAR01/17263
DOI URL: http://dx.doi.org/10.21474/IJAR01/17263

RESEARCH ARTICLE

STUDY OF COMMON FIXED POINT RESULT OF PAIR OF SELF MAPS IN G-METRIC SPACE.

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Manuscript Info

Manuscript History
Received: 15 May 2023
Final Accepted: 19 June 2023
Published: July 2023

Key words:-
G-Metric Space,G-Cauchy Sequence, Coincidence Point, Weakly Compatible Mapping

Abstract

In this article we study weakly compatible maps and proved common fixed point theorem of weakly compatible mappings in Generalized Metric space.
AMS Subject classification:-47H10,47H09.

Introduction:-

In 1922 S. Banach [1] provedContraction Principle in complete metric space.In 1976 G. Jungck [2] proved first common fixed point theorem for commuting maps in usual metric space.The concept of weak commutative maps which is a weaker type of commuting pair of maps was obtained by Sesa [3].

In 1986 Jungck[4] stated the concept of compatible mappings to generalize the concept of weak commutative pair of self maps .Then in 1986 Jungck [5] defined the concept of weakly compatible maps in Metric Space and proved some common fixed point theorems. In 2002, M. Aamri and D.E Moutawakil [6] defined the concept of E.A. Property for pair of self maps. In 1960 Gahler [7] derived a new metric space structure called as 2-Metric Space and claimed that this is more generalized structure of Metric Space.

In 1992 B.C. Dhage [9] introduced new generalized notion of metric space called as D-Metric Space .Mustafa Z. and Sims in 2003 [10] proved some of the results in D-metric Space are invalid.The concept of G-metric space was stated by Mustafa and Sims [11] and proved some results of fixed point in G-metric Space. In 2012 Zead Mustafa [12] proved some theorems of common fixed points for weakly compatible mappings.

Preliminaries

Definition 2.1 [11]. Let X be a nonempty set and $G: X \times X \times X \to R^+$ which satisfies the following axioms

(1) $G(a,b,c)=0$ if $a=b=c$ i.e. for every a,b,c in X coincides.

(2) $G(a,a,b)>0$ for every $a,b,c \in X$ s.t. $a \neq b$.

(3) $G(a,a,b) \leq G(a,b,c)\ \forall\ a,b,c \in X$

(4) $G(a,b,c)=G(b,a,c)=G(c,b,a)=$.......(Symmetry in all three variables)

(5) $G(a,b,c) \leq G(a,x,x)+G(x,b,c)$, for all a,b,c,x in X (rectangle inequality)

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Then the function G is said to be a generalized Metric Space or G-Metric on X and the pair (X,G) is called G-Metric Space.

Example 2.1 Let $G: X^3 \to R^+$ s.t. $G(a,b,c) =$ perimeter of the triangle with vertices at a,b,c in R^2, also by taking p in interior of the triangle then rectangle inequality is satisfied and the function G is a function on X.

Remark 2.1G-Metric Space is the generalization of the ordinary metric Space that is every G-metric space (X, G) gives ordinary metric space (X, d_G)

$d_G(a,b) = G(a,b,b) + G(a,a,b)$

Definition 2.3 Let (X,G) be a G-Metric space , let $\{a_n\}$ be a sequence of elements in X. The sequence $\{a_n\}$ is said to be G-convergent to a if $\lim_{m,n\to\infty} G(a,a_n,a_m) = 0$ i.e. for every ò >0 , there is N s.t. $G(a,a_n,a_m) <$ ò for all m,n $\geq N$. It is denoted as $\lim_{n\to\infty} a_n = a$.

Proposition 2.1 ([11]) If (X,G) be a G-Metric space. Then following are equivalent.

(i) $\{a_n\}$ is G-convergent to a.

(ii) $G(a_n,a_n,a) \to \infty$ as n $\to \infty$

(iii) $G(a_n,a,a) \to \infty$ as n $\to \infty$

(iv) $G(a_m,a_n,a) \to \infty$ as m, n $\to \infty$

Definition 2.4 Let (X,G) be a G-Metric space. A sequence $\{a_n\}$ is called G-Cauchy if ,

for ò>0 there is an N ò I^+ (set of positive Integers) s.t.

$G(a_n,a_m,a_l) <$ ò for all n,m,l $\geq N$

Proposition 2.3 ([11]) Let (X,G) be a G-Metric Space. Then for any a,b,c,x in X, it gives that

(i) If $G(a,b,c) = 0$ then $a = b = c$

(ii) $G(a,b,c) \leq G(a,a,b) + G(a,a,c)$

(iii) $G(a,b,b) \leq 2G(b,a,a)$

(iv) $G(a,b,c) \leq G(a,x,c) + G(x,b,c)$

(v) $G(a,b,c) \leq \frac{2}{3}(G(a,x,x) + G(b,x,x) + G(c,x,x))$

Definition 2.5 If S and T be self maps of a set X .If w=Sx=Tx for some x in X, then x is called coincidence point of S and T.

Definition 2.6 [5] Self maps S and T are said to be weakly compatible if they commute at their conicidence point i.e. if Sx=Tx for some x in X then $STx = TSx$

Example 2.4 Let X=[1, +∞) and G(a,b,c)=|a-b|+|b-c|+|a-c|.

Define S,T : X $\to$ X by $S(a) = 2a - 1$ and $T(a)=a^2$, a ò X , we say that a=1 is the only coincidence point and S(T(1))=S(1)=1 and

T(S(1))=T(1)=1 , so S and T are weakly compatible.

Main Result:-

Now we prove common fixed point theorem for the pair of weakly compatible maps for the new contraction.

Theorem 3.1:-Let (X,G) be a G-Metric Space which is Complete. If S and T be Weakly Compatible maps on X into itself, s.t.

(1) $S(X) \subseteq T(X)$

(2) $G(Sa,Sb,Sc) \le \alpha G(Sa,Tb,Tc) + \beta G(Ta,Sb,Tc) + \gamma G(Ta,Tb,Sc) +$

$\delta G(Sa,Tb,Tc)$, *for* all a,b,c in X & α, β, γ and $\delta \ge 0$

s.t. $0 \le \alpha + 3\beta + 3\gamma + \delta < 1$

(3)*Subspace* S(X) or T(X) is Complete. Then there exists a Unique Common fixed point of S and T in X

.

Proof:-Let us choose a_0 be an any element in X . Since $S(X) \subseteq T(X)$, we construct a sequence { b_n } in X s.t. for any a_1 in X , $Sa_0 = Ta_1$ .In general for a_{n+1} s.t.

$b_n = Sa_n = Ta_{n+1}$ for n=0,1,2...... From inequality (2) in hypothesis, we have

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le \alpha G(Sa_n, Ta_{n+1}, Ta_{n+1}) + \beta G(Ta_n, Sa_{n+1}, Ta_{n+1})$$
$$+\gamma G(Ta_n, Ta_{n+1}, Sa_{n+1}) + \delta G(Sa_n, Ta_{n+1}, Ta_{n+1})$$

$\therefore$ from the above sequence ,we have

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le \beta G(Sa_{n-1}, Sa_{n+1}, Sa_n) + \gamma G(Sa_{n-1}, Sa_n, Sa_{n+1})$$
$$(\because \alpha G(Sa_n, Sa_n, Sa_n) = 0 = \delta G(Sa_n, Sa_n, Sa_n))$$

$\because$ By symmetry, we have

$$G(Sa_{n-1}, Sa_{n+1}, Sa_n) = G(Sa_{n-1}, Sa_n, Sa_{n+1})$$
$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le (\beta + \gamma) G(Sa_{n-1}, Sa_n, Sa_{n+1}) \qquad (1.1)$$

By using rectangular inequality of G- metric space .We have

$$G(Sa_{n-1}, Sa_n, Sa_{n+1}) \le G(Sa_{n-1}, Sa_n, Sa_n) + G(Sa_n, Sa_{n+1}, Sa_n)$$
$$\le G(Sa_{n-1}, Sa_n, Sa_n) + 2G(Sa_n, Sa_{n+1}, Sa_{n+1})$$

($\because$ By using Proposition 2.1) from given hypothesis (ii), we have

$$(1 - 2\beta - 2\gamma) G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le (\beta + \gamma) G(Sa_{n-1}, Sa_n, Sa_n)$$
$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le \frac{\beta + \gamma}{1 - 2\beta - 2\gamma} G(Sa_{n-1}, Sa_n, Sa_n)$$
$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le q_1 G(Sa_{n-1}, Sa_n, Sa_n)$$

Where $q_1 = \dfrac{\beta + \gamma}{1 - 2\beta - 2\gamma} < 1$

By continuing in this way,we get

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le q_1^{\,n} G(Sa_0, Sa_1, Sa_1) \qquad (1.2)$$

For all n, m $\in I^+$, Let m > n and by using rectangle inequality

Consider

$$G(b_n,b_m,b_m) \le G(b_n,b_{n+1},b_{n+1}) + G(b_{n+1},b_{n+2},b_{n+2})$$
$$+........................+G(b_{m-1},b_m,b_m)$$
$$G(b_n,b_m,b_m) \le (q_1^n + q_1^{n+1} + ... + q_1^{m-1})G(b_0,b_1,b_1)$$
$$(\therefore \text{ by using (2)})$$
$$\le \frac{q_1^n}{1-q_1} G(b_0,b_1,b_1)$$

As $n,m \to \infty \therefore$ R.H.S. of above inequality tends to 0. We have $lim_{n\to\infty} G(b_n,b_m,b_m) = 0 \therefore$ the sequence { b_n } is a G-Cauchy sequence in X . Since $S(X)$ or $T(X)$ is Complete subspace of X then subsequence of { b_n } must get a limit in $T(X)$.

$\therefore$ The Sequence { b_n } also convergent .Since { b_n } Contains a Convergent subsequence in $T(X)$. Say it c_1 . Let $u = Tc^{-1}$ then Tu=c_1 Now we prove that Su=c_1

On putting $a = u, b = a_n$ and c=a_n in (ii), We have

$$G(Su,Sa_n,Sa_n) \le \alpha G(Su,Ta_n,Ta_n) + \beta G(Tu,Sa_n,Ta_n) + \gamma G(Tu,Ta_n,Sa_n)$$
$$+ \delta G(Su,Ta_n,Ta_n)$$

as $n \to \infty$,above inequality becomes

$\beta G(Tu,Sa_n,Ta_n) = \beta G(c_1,c_1,c_1) = 0$ also

$\gamma G(Tu,Ta_n,Sa_n) = G(c_1,c_1,c_1) = 0$

$\therefore$ We have

$G(Su,c_1,c_1) \le \alpha G(Su,c_1,c_1)$

This gives , $Su = c_1$

$\therefore Su = Tu = c_1$ $\therefore$ u is a coincident point of S and T.

As S and T are weakly Compatible $\therefore$ By definition $STu = TSu \therefore Sc_1 = Tc_1$

Now we show that $Sc_1 = c_1$. Suppose $Sc_1 \ne c_1$,

$\therefore G(Sc_1,c_1,c_1) > 0$ In (ii) putting a=$c_1, b = u, c = u$

$\therefore$ We have

$$G(Sc_1,c_1,c_1) = G(Sc_1,Su,Su)$$
$$\le \alpha G(Sc_1,Tu,Tu) + \beta G(Tc_1,Su,Tu)$$
$$+\gamma G(Tc_1,Tu,Su) + \delta G(Sc_1,Tu,Tu)$$
$$=(\alpha + \beta + \gamma + \delta)G(Sc_1,c_1,c_1)$$
$$<G(Sc_1,c_1,c_1)$$

Which is a contradiction. $\therefore$ this gives $Sc_1 = c_1$

$\therefore Sc_1 = Tc_1 = c_1$ $\therefore$ c_1 is a Common fixed point of S and T .

To prove Uniqueness,
Suppose that c' is another Common fixed Point of S and T which is distinct from c_1. i.e. $c_1 \neq c'$.
Consider,

$$G(c_1, c', c') = G(Sc_1, Sc', Sc')$$
$$\leq \alpha G(Sc_1, Tc', Tc') + \beta G(Tc_1, Sc', Tc')$$
$$+\gamma G(Tc_1, Tc', Sc') + \delta G(Sc_1, Tc', Tc')$$
$$=(\alpha + \beta + \gamma + \delta) G(c_1, c', c')$$
$$< G(c_1, c', c')$$

$\therefore c_1 = c'$
Hence proof.

Conclusion:-
Thus we have proved Common fixed theorem for pair of weakly compatible mappings.

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