ISSN(O): 2320-5407 Int. J. Adv. Res. 13(05), May-25, 280-287

Journal Homepage: - www.journalijar.com

INTERNATIONAL JOURNAL OF ADVANCED RESEARCH (IJAR)

ISSN NO. 2320-5407

Article DOI: 10.21474/IJAR01/20900

DOI URL: http://dx.doi.org/10.21474/IJAR01/20900

RESEARCH ARTICLE

LEAST SQUARES ESTIMATORS OF DRIFT PARAMETER FOR DISCRETELY OBSERVED FRACTIONAL VASICEK-TYPE MODEL

Maoudo Faramba Balde , Mamadou Saliou Bah, Bakary Kourouma and Abdoulaye Mendy

1. Gamal Abdel Nasser University of Conakry, Department of Mathematics, B.P. 1147, Conakry, Guinea

Manuscript Info

Manuscript History

Received: 24 March 2025

Final Accepted: 27 April 2025

Published:May 2025

Key words:-

Fractional Brownian motion; Vasicek-type model; Young integral; Parameter estimation; Discrete observations; Tightness.

Abstract

We study the drift parameter estimation problem for a fractional Vasicek-typemodel $X := \{X_t, t \geqslant 0\}$, that is defined as $dX_t = \theta(\mu + X_t)dt + dB_t^H$, $t \geqslant 0$ withunknown parameters θ>0 and $\mu \in \mathbb{R}$, where $\{B_t^H, t \geqslant 0\}$is a fractional Brownianmotion of Hurst index H ∈]0, 1[. Let $\widehat{\theta_t}$and $\widehat{\mu_t}$be the least squares-type estimatorsof θand μ, respectively, based on continuous observation of X. In this paper weassume that the process $\{X_t, t \geqslant 0\}$is observed at discrete time instants $t_i = i\Delta_n$,i=1,...,n. We analyze discrete versions $\widetilde{\theta_n}$and $\widetilde{\mu_n}$for $\widehat{\theta_t}$and $\widehat{\mu_t}$ respectively. We show that the sequence $\sqrt{n\Delta_n}\left(\widetilde{\theta_n} - \theta\right)$ is tight and $\sqrt{n\Delta_n}(\widetilde{\mu_n} - \mu)$ is not tight. Moreover, we prove the stronge consistency of $\widetilde{\theta_n}$.

Introduction:-

2010 AMS Classification Numbers: 60G15; 60G22; 62F12; 62M09; 62M86.

1. Introduction

Let $B^H := \{B_t^H, t \geqslant 0\}$ bea fractional Brownian motion (fBm in short) of Hurst index$H \in]0,1[$, that is, a centered Gaussian process starting from zero withcovariance

$$E(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

Notice that when $H = \frac{1}{2}$, $B^{\frac{1}{2}}$is a standard Brownian motion.

Consider the fractional Vasicek-type of the first kind $X := \{X_t, t \geqslant 0\}$, defined as the unique (pathwise) solution to

$$\begin{cases} dX_t = \theta(\mu + X_t)dt + dB_t^H, & t > 0, \\ X_0 = 0, \end{cases} \qquad (1.1)$$

where $\mu \in \mathbb{R}$and $\theta > 0$are considered as unknown parameters.

Let $\widehat{\theta_T}$ and $\widehat{\mu_T}$ bethe least squares-type estimators of and μ, respectively, based oncontinuous observation of X. It is well known that, least squares estimators method are motivated by the argument of minimize a quadratic function μa and θ, respectively,

$(\mu, \theta) \mapsto \int_0^T |\dot{X}_t - \theta(\mu + X_t)|^2 \, dt$

Corresponding Author:- Maoudo Faramba Baldé

Address:- Gamal Abdel Nasser University of Conakry, Department of Mathematics, B.P. 1147,

where $\dot{X}_t$ denotes the differentiation of X_t with respect to t. By taking the partial derivative for μa and θ, separately, and then solving the equations, we can obtain the least squares estimators of μa and θ, denoted by $\widehat{\theta_T}$ and $\widehat{\mu_T}$ respectively,

$$\widehat{\theta_T} = \frac{\frac{1}{2}TX_T^2 - X_T\int_0^T X_s ds}{T\int_0^T X_S^2 ds - \left(\int_0^T X_s ds\right)^2} \quad (1.2)$$

$$\widehat{\mu_T} = \frac{\int_0^T X_S^2 ds - \frac{1}{2}X_T\int_0^T X_S ds}{\frac{1}{2}TX_T - \int_0^T X_s ds} \quad (1.3)$$

The study of various problems related to model (1.1) has gained attention in recent years. In finance modeling μcan be interpreted as the long-run equilibri-um value of Xwhereas θrepresents the speed of reversion. For a motivation in mathematical finance and further references, we refer the reader to [2,3, 4, 5]. When $B^H$is replaced by a standard Brownian motion, the model (1.1) with μ = 0 was originally proposed by Ornstein and Uhlenbeck and then it was generalized by Vasicek (see [14]).Recent works [8], [11] and [15] developed statistical inference for several fractional Ornstein-Uhlenbeck (fOU in short) process in the ergodic case . The case of non-ergodic fOU process is presented in [1], [6], [7], [9] and [10].

Let us describe what is known about the asymptotic behaviors of the estimators (1.2) and (1.3), studied in [9]:

- for every$H \in (0,1)$, we have almost surely, as$T \to \infty$,

$$\left(\widehat{\theta_T}, \widehat{\mu_T}\right) \to (\theta, \mu) \quad (1.4)$$

- suppose that $H \in (0,1)$, and$N_1 \sim \mathcal{N}(0,1)$, $N_2 \sim \mathcal{N}(0,1)$, and B^H are independent, then as $T \to \infty$,

$$\left(e^{\theta T}\left(\widehat{\theta_T} - \theta\right), T^{1-H}\left(\widehat{\mu_T} - \mu\right)\right) \xrightarrow{\text{Law}} \left(\frac{2\theta\sigma_{B^H} N_2}{\mu + \zeta_{B^H,\infty}}, \frac{1}{\theta}N_1\right), \quad (1.5)$$

$\sigma_{B^H}^2 = \frac{H\Gamma(2H)}{\theta^{2H}}$, and $\zeta_{B^H,\infty} \sim \mathcal{N}\left(0, \sigma_{B^H}^2\right)$ is independent of N_1 and N_2.

From a practical standpoint, in parametric inference, it is more realistic and interesting to consider asymptotic estimation for (1.1) based on discrete observations. Then, in the present paper, we will assume that the process X given in (1.1) is observed equidistantly in time with the step sizeΔ_n:$t_i=i\Delta_n$,i=1,...,n and $T_n = n\Delta_n$denotes the length of the"observation window".

Here, based on discrete-time observations of X defined in (1.1), we will analyse thefollowing discrete versions $\widetilde{\theta_n}$and $\widetilde{\mu_n}$for $\widehat{\theta_t}$and $\widehat{\mu_t}$ respectively, defined as

$$\widetilde{\theta_n} = \frac{\frac{1}{2}X_{T_n}^2 - \frac{X_{T_n}}{n}\sum_{i=1}^n X_{t_{i-1}}}{\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2 - \frac{\Delta_n}{n}\left(\sum_{i=1}^n X_{t_{i-1}}\right)^2} \quad (1.6)$$

$$\widetilde{\mu_n} = \frac{\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2 - \frac{1}{2}X_{t_n}\Delta_n \sum_{i=1}^n X_{t_{i-1}}}{\frac{1}{2}T_n X_{T_n} - \Delta_n \sum_{i=1}^n X_{t_{i-1}}} \quad (1.7)$$

Our paper is organized as follows. In Section 2, we give the basic knowledge about Young integral and some preliminary results, which will be very useful to our main proof. In Section 3, based on discrete observations of X defined in (1.1), we study the rate consistency of the estimators $\widetilde{\theta_n}$and $\widetilde{\mu_n}$.

2. Preliminaries

In this section, we briefly recall some basic elements of Young integral (see [16]), which are helpful for some of the arguments we use.

For any$\alpha \in [0,1]$,, we denote by$\mathcal{H}^\alpha([0,1])$the set of Holder continuous functions, that is, the set of functions $f: [0,T] \to \mathbb{R}$such that

$$|f|_\alpha = \sup_{0 \le s < t \le T} \frac{|f(t) - f(s)|}{(t-s)^\alpha} < \infty.$$

We also set $|f|_\infty := \sup_{t\in[0,T]}|f(t)|$ and equip$\mathcal{H}^\alpha(|[0,T]|)$ with the norm

$$\|f\|_\alpha := |f|_\alpha + |f|_\infty.$$

Let $f \in \mathcal{H}^\alpha([0,T])$, and consider the operator $T_f: C^1([0,T]) \to C^0([0,T])$ defined as

$$T_f(g)(t) = \int_0^t f(u)g'(u)du, \ t \in [0,T].$$

It can be shown (see, [13]) that, for any $\beta \in]1-\alpha,1[$, there exists a constant $C_{\alpha,\beta,T} > 0$ depending only on α,β and T such that, for any $g \in \mathcal{H}^{\alpha}([0,T])$,
$\left\|\int_0^{\cdot} f(u)g'(u)du\right\|_{\beta} \leq C_{\alpha,\beta,T}\|f\|_{\alpha}\|g\|_{\beta}$.
We deduce that, for any $\alpha \in]0,1[$ any $f \in \mathcal{H}^{\alpha}([0,T])$ and any $\beta \in]1-\alpha,1[$ the linear operator $T_f: C^1([0,T]) \subset \mathcal{H}^{\beta}([0,T]) \to \mathcal{H}^{\beta}([0,T])$, defined as $T_f(g) = \int_0^{\cdot} f(u)g'(u)du$ is continuous with respect to the norm $\|.\|_{\beta}$.
By density, it extends (in an unique way) to an operator defined on $\mathcal{H}^{\beta}$. As consequence, if $f \in \mathcal{H}^{\alpha}(|[0,T]|)$, if $g \in \mathcal{H}^{\beta}(|[0,T]|)$ and if $\alpha+\beta > 1$ then the (so-called) Young integral $\int_0^{\cdot} f(u)dg(u)$ is (well) defined as being $T_f(g)$.
The Young integral satisfies the following formula. Let $f \in \mathcal{H}^{\alpha}([0,T])$ with
$\alpha \in]0,1[$ and $g \in \mathcal{H}^{\beta}([0,T])$ with $\beta \in]0,1[$ such that $\alpha+\beta > 1$. Then $\int_0^{\cdot} f_u dg_u$ and $\int_0^{\cdot} f_u dg_u$ are well-defined as Young integrals. Moreover, for all $t \in [0,T]$,

$$f_t g_t = f_0 g_0 + \int_0^t g_u\, df_u + \int_0^t f_u dg_u\,. \qquad (2.1)$$

In order to study the strong consistency, we will need the following direct consequence of the BorelCantelli Lemma (see Kloeden and Neuenirch (2007)), which allows us to turn convergence rates in the p-th mean into pathwise convergence rates.
Lemma 2.1. ([12]) Let $\beta > 0$ and let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of random variables. If for every $p \geq 1$ there exists a constant $c_p > 0$ such that for all $n \in \mathbb{N}$,

$$(E|Z_n|^p)^{1/p} \leq C_p.n^{-\beta},$$

then for all $\varepsilon > 0$ there exists a random variable η_{ε} such that
$|Z_n| \leq \eta_{\varepsilon}.n^{-\beta+\varepsilon}$ almost surely
for all $n \in \mathbb{N}$. Moreover, $E|\eta_{\varepsilon}|^p < \infty$ for all $p \geq 1$.
Next, let us note that the unique solution to (1.1) can be written as

$$X_t = \mu(e^{\theta t}-1) + e^{\theta t}\int_0^t e^{-\theta s}\, dB_s^H\ ,\ \ t \geq 0\,. \qquad (2.2)$$

We will also need the following processes, for every $t \geq 0$
$\zeta_t := \int_0^t e^{-\theta t} dB_s^H$; $\sum_t := \int_0^t X_s ds Z_t := \int_0^t e^{-\theta t} B_s^H ds$ $\qquad (2.3)$
Using (2.2), we can write

$$X_t = u(e^{-\theta t}-1) + e^{-\theta t}\zeta_t\,. \qquad (2.4)$$

Furthermore, by (1.1),

$$X_t = \mu\theta t + B_t^H\,. \qquad (2.5)$$

Moreover, applying the formula (2.1), we have

$$\zeta_t = e^{-\theta t}B_t^H + \theta\int_0^t e^{-\theta t}B_s^H ds = e^{-\theta t}B_t^H + \theta Z_t\,. \qquad (2.6)$$

From (2.4) we can also write
$X_t = e^{\theta t}Z_t$, With $Z_t = \mu(1-e^{-\theta t}) + \zeta_t$ $\quad t \geq 0.$ $\qquad (2.7)$
Lemma 2.2.([6]). Assume that the process B^H has Hölder continuous path of order $\gamma \in]0,1[$. Let ζ be given by (2.3). Then for all $\varepsilon \in]0,\gamma[$ the process ζ admits a modification with $(\gamma-\varepsilon)$-Hölder continuous paths.
Moreover
$Z_t \to Z_\infty := \int_0^\infty e^{-\theta t}B_s^H ds$, $\quad \zeta_t \to \zeta_\infty := \theta Z_\infty$ $\qquad (2.8)$
almost surely and in $L^2(\Omega)$ as $T \to \infty$.
Lemma3.2. ([9]). Assume that H ∈ (0, 1). Then, almost surely, as

$$e^{-\theta T}X_T \to \mu + \zeta_\infty \qquad (2.9)$$

$$e^{-\theta T}\int_0^T X_s\, ds \to \frac{1}{\theta}(\mu+\zeta_\infty) \qquad (2.10)$$

$$\frac{e^{-\theta T}}{T}\int_0^T sX_s\, ds \to \frac{1}{\theta}(\mu+\zeta_\infty) \qquad (2.11)$$

$\frac{e^{-\theta T}}{T^{\delta}}\int_0^T |X_s|\, ds ds \to 0$ for any δ>0 $\qquad (2.12)$

$$e^{-2\theta T}\int_0^T X_s^2\, ds \to \frac{1}{2\theta}(\mu+\zeta_\infty)^2 \qquad (2.13)$$

where is defined in Lemma 2.2.

From now on, the generic constant is always denoted by C(.) which depends on certain parameters in the parentheses.

3.Main results

Lemme 3.1. Let $(S_n, n \geq 1)$ and $(R_n, n \geq 2)$ be a random sequences defined by

$S_n := \Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2$; $S_n := \Delta_n \sum_{i=1}^{n-1} e^{-2\theta(T_n - t_i)}(Z_{t_i}^2 - Z_{t_{i-1}}^2)$. (3.1)

Then for every $n \geq 2$,

$S_n e^{-2\theta T_n} = \frac{\Delta_n}{e^{2\Delta_n - 1}}(Z_{t_{n-1}}^2 - R_n)$. (3.2)

In addition if $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to \infty$for some $\alpha > 0$,

$R_n \to 0$ almost surely as $n \to \infty$. (3.3)

In particular,

$S_n e^{-2\theta T_n} \to \frac{(\mu + \zeta_\infty)^2}{2\theta}$ almost surely as $n \to \infty$. (3.4)

Proof. Using (2.7), we can write for every n ≥ 2,

$$S_n e^{-2\theta T_n} = \Delta_n \sum_{i=1}^{n} e^{-2\theta(n-i)\Delta_n} e^{-2\theta\Delta_n} Z_{t_{i-1}}^2$$
$$= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \sum_{i=1}^{n} e^{-2\theta(n-i)\Delta_n} \left(1 - \frac{1}{e^{2\theta\Delta_n}}\right) Z_{t_{i-1}}^2.$$

This imply that

$$S_n e^{-2\theta T_n} = \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \sum_{i=1}^{n} \left(e^{-2\theta(n-i)\Delta_n} - e^{-2\theta(n+1-i)\Delta_n}\right) Z_{t_{i-1}}^2$$
$$= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1}\left[Z_{t_{n-1}}^2 - \sum_{i=1}^{n} (Z_{t_{i-1}}^2 - Z_{t_{i-2}}^2) e^{-2\theta(n+1-i)\Delta_n}\right]$$
$$= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1}\left[Z_{t_{n-1}}^2 - R_n\right] ,$$

which implies (3.2).

Let us now prove (3.3). First, observe that $\Delta_n \to 0$and $n\Delta_n^{1+\alpha} \to \infty$imply that $n\Delta_n \to \infty$. On the other hand, (2.8) implies

$$Z_T \to \mu + \zeta_\infty \quad (3.5)$$

almost surely and in $L^2(\Omega)$ as $T \to \infty$.

Thus, by using (2.7),$\{\zeta_t, t \geq 0\}$ is Gaussian and (3.5), we obtain for every $p \geq 0$,

$$\left(E\left[\left|Z_{t_i}^2 - Z_{t_{i-1}}^2\right|^p\right]\right)^{\frac{1}{p}} \leq \left(E\left[\left|(Z_{t_i} - Z_{t_{i-1}})(Z_{t_i} + Z_{t_{i-1}})\right|^p\right]\right)^{\frac{1}{p}}$$
$$\leq C(\mu, \theta, H)\left(E\left[\left|Z_{t_i} - Z_{t_{i-1}}\right|^p\right]\right)^{\frac{1}{p}}$$
$$\leq C(\mu, \theta, H)\left(\left|e^{-\theta t_i} - e^{-\theta t_{i-1}}\right| + \left(E\left[\left|\zeta_{t_i} - \zeta_{t_{i-1}}\right|^p\right]\right)^{\frac{1}{p}}\right)$$
$$\leq C(p, \mu, \theta, H)\left(e^{-\theta t_i}\left|e^{\theta\Delta_n} - 1\right| + \left(E\left[\left|\zeta_{t_i} - \zeta_{t_{i-1}}\right|^2\right]\right)^{\frac{1}{2}}\right)$$
$$\leq C(p, \mu, \theta, H)(\Delta_n e^{-\theta t_i} + \Delta_n^H e^{-\theta i\Delta_n})$$
$$\leq C(p, \mu, \theta, H)\Delta_n^H e^{-\theta t_i}, \quad (3.1)$$

where we used $\frac{e^{\theta\Delta_n} - 1}{\Delta_n} \to 0$ and the following inequality given in [10] for every $i = 1, \ldots, n$, $n \geq 1$,

$$\left(E\left[\left|\zeta_{t_i} - \zeta_{t_{i-1}}\right|^2\right]\right)^{\frac{1}{2}} \leq C(\theta, H)\Delta_n^H e^{-\theta t_i} \quad .$$

Thus for every $p \geq 1$,

$$(E[|R_n|^p])^{\frac{1}{p}} \leq \sum_{i=1}^{n-1} e^{-2\theta(n-i)\Delta_n} \left(E\left[\left|Z_{t_i}^2 - Z_{t_{i-1}}^2\right|^p\right]\right)^{\frac{1}{p}}$$
$$\leq C(p, \mu, \theta, H)e^{-\theta n\Delta_n}\Delta_n^H \sum_{i=1}^{n-1} e^{-\theta(n-i)\Delta_n}$$

$$\leq C(p,\mu,\theta,H)e^{-\theta n\Delta_n}\Delta_n^H e^{-\theta\Delta_n}\frac{1-e^{-\theta(n-1)\Delta_n}}{1-e^{-\theta\Delta_n}}$$
$$\leq C(p,\mu,\theta,H)\Delta_n^{H-1}e^{-\theta n\Delta_n} \quad . \quad (3.7)$$

The last inequality comes from $\Delta_n \to 0$ and $\frac{\Delta_n}{1-e^{-\theta\Delta_n}} \to \frac{1}{\theta}$.
Taking a constant β verifying $\frac{1-\gamma}{\beta} < \alpha < \beta$, there is $\varepsilon > 0$ such that $\alpha = \frac{\varepsilon+1-\gamma}{\beta-\varepsilon}$.
Hence, we can write

$$(n\Delta_n)^\beta \Delta_n^{1-\gamma} = n^\varepsilon (n\Delta_n^{1+\alpha})^{\beta-\varepsilon} \quad . \quad (3.8)$$

As a consequence, by (3.7) and (3.8),

$$(E[|R_n|^p])^{\frac{1}{p}} \leq C(p,\theta,\mu,H)\Delta_n^{\gamma-1}e^{-\theta n\Delta_n}$$
$$\leq C(p,\theta,\mu,H)\frac{1}{n^\varepsilon (n\Delta_n^{1+\alpha})^{\beta-\varepsilon}}\frac{(n\Delta_n)^\beta}{e^{\theta n\Delta_n}}$$
$$\leq C(p,\theta,\mu,H)n^{-\varepsilon}. \quad (3.9)$$

Therefore, by combining (3.9) and Lemma 2.1, the convergence (3.3) is proved.
On the other hand, the convergence (3.4) is a direct consequence of (3.2), (3.3) and (3.5).□

Lemme 3.2. Define for every $n \geq 1$

$$D_n := \frac{e^{-2\theta T_n}}{n}\sum_{i=1}^n X_{t_{i-1}}. \quad (3.10)$$

Assume that $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to \infty$ for some $\alpha > 0$, then, for every $n \geq 1$,

$$E(D_n{}^2) \leq C(\theta,\mu,H,\alpha)n^{-\frac{2\alpha}{1+\alpha}} \quad (3.11)$$

Moreover, for every $0 \leq \delta < 1$,

$$E\left[\left((n\Delta_n)^\delta D_n\right)^2\right] \leq \mathrm{C}(\theta,\mu,H,\alpha)n^{-\frac{2\alpha(1-H)}{1+\alpha}} . \quad (3.12)$$

As a consequence, for every $0 \leq \delta < 1$,

$$(n\Delta_n)^\delta \to 0 \ \ \textit{almost surely as}\ n \to \infty. \quad (3.13)$$

Proof. We first prove (3.11). Using (2.7) and (3.5), we have

$$E(D_n^2) = \frac{e^{-2\theta T_n}}{n^2}\sum_{i,j=1}^n E\left(X_{t_{i-1}}X_{t_{j-1}}\right) = \frac{e^{-2\theta T_n}}{n^2}\sum_{i,j=1}^n e^{\theta t_{i-1}+\theta t_{j-1}}E\left(Z_{t_{i-1}}Z_{t_{j-1}}\right)$$
$$\leq C(\theta,\mu,H)\frac{e^{-2\theta T_n}}{n^2}\sum_{i,j=1}^n e^{\theta t_{i-1}+\theta t_{j-1}} = C(\theta,\mu,H)\left(\frac{e^{-\theta T_n}}{n}\sum_{i=1}^n e^{\theta t_{i-1}}\right)^2$$
$$= C(\theta,\mu,H)\left(\frac{e^{-\theta T_n}}{n}\frac{e^{\theta n\Delta_n}-1}{e^{\theta\Delta_n}-1}\right)^2$$
$$\leq C(\theta,\mu,H)\left(\frac{1}{n\Delta_n}\frac{\Delta_n}{e^{\theta\Delta_n}-1}\right)^2$$
$$\leq C(\theta,\mu,H)\frac{1}{(n\Delta_n)^2} \quad . \quad (3.14)$$

Setting $\gamma = \frac{\alpha}{1+\alpha}$, we obtain

$$E(D_n^2) \leq C(\theta,\mu,H)\frac{n^{-2\gamma}}{(n^{1-\gamma}\Delta_n)^2} = C(\theta,\mu,H)\frac{n^{-\frac{2\alpha}{1+\alpha}}}{(n\Delta_n^{1+\alpha})^{\frac{1}{1+\alpha}}} \leq C(\theta,\mu,H,\alpha)n^{-\frac{2\alpha}{1+\alpha}},$$

which proves (3.11).
For (3.12), by (3.14), we have,

$$E[((n\Delta_n)^H D_n)^2] \leq C(\theta,\mu,H)(n\Delta_n)^{-2(1-\gamma)} .$$

Thus, using similar arguments as in (3.8), we can conclude

$$E[((n\Delta_n)^H D_n)^2] \leq C(\theta,\mu,H,\alpha)n^{-\frac{2\alpha(1-H)}{1+\alpha}} ,$$

which implies the desired result.
Finally, the convergence (3.13) is a direct consequence of (3.12) and Lemma 2.1. □

Definition 3.1. Let $\{Z_n\}$ be a sequence of random variables defined on $(\Omega,\mathcal{F},P)$. We say $\{Z_n\}$ is tight (or bounded in probability), if for every $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that,

$P(|Z_n| > M_\varepsilon) < \varepsilon$, for all n.
Theorem 3.3. Let $H \in (0,1)$. Suppose that $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to \infty$for some $\alpha > 0$. Then, for every $q \geq 1$,
$\Delta_n^q e^{\theta T_n}(\widetilde{\theta_n} - \theta)$ is not tight. (3.15)
In addition if we assume that $n\Delta_n^3 \to 0$ as $n \to \infty$, then the estimator $\widetilde{\theta_n}$ is
$\sqrt{T_n}$-consistent in the sens that the sequence
$\sqrt{T_n}(\widetilde{\theta_n} - \theta)$ is tight (3.16)
and
$\sqrt{T_n}(\widetilde{\mu_n} - \mu)$ is not tight. (3.17)
Proof. Fix $q \geq 1$. From (1.6) and (2.7) we can write

$$\Delta_n^q e^{\theta T_n}(\widetilde{\theta_n} - \theta)$$

$$= \Delta_n^q e^{\theta T_n}\left(\frac{\frac{1}{2}Z_{T_n}^2 - Z_{T_n}D_n}{e^{2\theta T_n}S_n - \left(\sqrt{T_n}D_n\right)^2} - \theta\right)$$

$$= \frac{\Delta_n^q e^{\theta T_n}}{2e^{2\theta T_n}S_n - 2\left(\sqrt{T_n}D_n\right)^2}\left[\left(Z_{T_n}^2 - Z_{T_{n-1}}^2\right) + \left(1 - \frac{2\theta\Delta_n}{e^{2\theta\Delta_n} - 1}\right)Z_{T_{n-1}}^2 - 2\theta\left(e^{-2\theta T_n}S_n - \frac{\Delta_n}{e^{2\theta\Delta_n} - 1}Z_{T_{n-1}}^2\right)\right]$$

Moeover,

$$e^{-2\theta T_n}S_n - \frac{\Delta_n}{e^{2\theta\Delta_n} - 1}Z_{T_{n-1}}^2 = e^{-2\theta T_n}\Delta_n\sum_{i=1}^{n} e^{2\theta t_{i-1}}Z_{t_{i-1}}^2 - \frac{\Delta_n}{e^{2\theta\Delta_n} - 1}Z_{T_{n-1}}^2$$

$$= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1}\left(\sum_{i=1}^{n} e^{-2\theta(T_n - t_i)}Z_{t_{i-1}}^2 - \sum_{i=1}^{n} e^{-2\theta(T_n - t_{i-1})}Z_{t_{i-1}}^2 - Z_{T_{n-1}}^2\right)$$

$$= \frac{\Delta_n}{e^{2\theta\Delta_n - 1}}R_n ,$$

where R_n is given by (3.1).
Thus we obtain

$$\Delta_n^q e^{\theta T_n}(\widetilde{\theta_n} - \theta)$$

$$= \frac{\Delta_n^q e^{\theta T_n}}{2e^{2\theta T_n}S_n}\left[\left(Z_{T_n}^2 - Z_{T_{n-1}}^2\right) + \left(1 - \frac{2\theta\Delta_n}{e^{2\theta\Delta_n} - 1}\right)Z_{T_{n-1}}^2 + \left(\frac{2\theta\Delta_n}{e^{2\theta\Delta_n} - 1}\right)R_n\right]. (3.18)$$

According to (3.6), we get

$$\left(E\left[\left(\Delta_n^q e^{\theta T_n}\left(Z_{T_n}^2 - Z_{T_{n-1}}^2\right)\right)^2\right]\right)^{\frac{1}{2}} \leq C(\theta,\mu,H)\Delta_n^{q+H} \to 0 . \qquad (3.19)$$

We also have

$$\Delta_n^q e^{\theta T_n}\left(1 - \frac{2\theta\Delta_n}{e^{2\theta\Delta_n} - 1}\right) = \Delta_n^{q+1}e^{\theta T_n}\left(\frac{e^{2\theta\Delta_n} - 1 - 2\theta\Delta_n}{\Delta_n^2}\frac{\Delta_n}{e^{2\theta\Delta_n} - 1}\right) \to \infty \qquad (3.20)$$

since

$$\Delta_n^{q+1}e^{\theta T_n} = \left(n\Delta_n^{q+\alpha}\right)^{\frac{q+1}{\alpha}}\frac{e^{\theta T_n}}{T_n^{\frac{q+1}{\alpha}}} \to \infty \text{ and } \left(\frac{e^{2\theta\Delta_n} - 1 - 2\theta\Delta_n}{\Delta_n^2}\frac{\Delta_n}{e^{2\theta\Delta_n} - 1}\right) \to \theta.$$

Furthermore, by (3.7),

$$\left(E\left[\left(\Delta_n^q e^{\theta T_n}R_n\right)^2\right]\right)^{\frac{1}{2}} \leq C(\theta,\mu,H)\Delta_n^{q+H-1} \to 0 . \qquad (3.21)$$

Combining (3.18), (3.19), (3.20), (3.21) and (3.4), we conclude that for every $q \geq 1$,$\Delta_n^q e^{\theta T_n}(\widetilde{\theta_n} - \theta)$ is not tight.
For $0 \leq q < 1$ we have

$$\Delta_n^q e^{\theta T_n}(\widetilde{\theta_n} - \theta) = \Delta_n^{q-1}\left(\Delta_n e^{\theta T_n}(\widetilde{\theta_n} - \theta)\right),$$

which completes the proof of (3.15), where we used the previous case and the fact that $\Delta_n^{q-1} \to \infty$.
Next, let us prove (3.16). It follows from (3.18) that

$$\sqrt{T_n}(\widetilde{\theta_n} - \theta) = \frac{\sqrt{T_n}}{2e^{-2\theta T_n}S_n}\left[\left(Z_{T_n}^2 - Z_{T_{n-1}}^2\right) + \left(1 - \frac{2\theta\Delta_n}{e^{2\theta\Delta_n - 1}}\right)Z_{T_{n-1}}^2 + \left(\frac{2\theta\Delta_n}{e^{2\theta\Delta_n - 1}}\right)R_n\right].$$

Combining this with

$$\left(E\left[\left(\sqrt{T_n}\left(Z_{T_n}^2 - Z_{T_{n-1}}^2\right)\right)^2\right]\right)^{\frac{1}{2}} \le C(\theta,\gamma)\Delta_n^{\gamma}\sqrt{T_n}e^{-\theta T_n} \to 0 ,$$

$$\sqrt{T_n}\left(1 - \frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right) = \sqrt{n\Delta_n^3}\left(\frac{e^{2\theta\Delta_n}-1-2\theta\Delta_n}{\Delta_n^2}\frac{\Delta_n}{e^{2\theta\Delta_n}-1}\right) \to 0 ,$$

$$\left(E\left[\left(\sqrt{T_n}R_n\right)^2\right]\right)^{\frac{1}{2}} \le C(\theta,\gamma)\Delta_n^{\gamma-1}\sqrt{T_n}e^{-\theta T_n} = C(\theta,\gamma)\frac{T_n^{\frac{1}{2}+\frac{1-\gamma}{\alpha}}e^{-\theta T_n}}{(n\Delta_n^{1+\alpha})^{\frac{1-\gamma}{\alpha}}} \to 0 ,$$

and the convergence (3.4), we deduce that

$$\sqrt{T_n}\left(\widetilde{\theta_n} - \theta\right) \to 0 \qquad (3.22)$$

in probability, which proves (3.16).

Now it remains to prove (3.17). Using (1.6) and (1.7), we can show that $\widetilde{\theta_n}$ and $\widetilde{\mu_n}$ satisfy

$$\widetilde{\theta_n}\widetilde{\mu_n}T_n = \frac{X_{T_n}\left(\sum_{i=1}^n X_{t_{i-1}}^2 - \frac{X_{T_n}}{n}\sum_{i=1}^n X_{t_{i-1}}\right)}{\sum_{i=1}^n X_{t_{i-1}}^2 - \frac{1}{n}\left(\sum_{i=1}^n X_{t_{i-1}}\right)^2}$$

$$= X_{T_n} - \widetilde{\theta_n}\Delta_n\sum_{i=1}^n X_{t_{i-1}}.$$

Combining this with (1.1), we obtain

$$T_n\widetilde{\theta_n}(\widetilde{\mu_n} - \mu)$$

$$= \mu T_n\left(\theta - \widetilde{\theta_n}\right) + \theta\int_0^{T_n} X_t\,dt + B_{T_n}^H - \widetilde{\theta_n}\Delta_n\sum_{i=1}^n X_{t_{i-1}}$$

$$= \mu T_n\left(\theta - \widetilde{\theta_n}\right) + \widetilde{\theta_n}\left(\int_0^{T_n} X_t\,dt - \Delta_n\sum_{i=1}^n X_{t_{i-1}}\right) + \left(\theta - \widetilde{\theta_n}\right)\int_0^{T_n} X_t\,dt + B_{T_n}^H.$$

Thus, we obtain

$$\sqrt{T_n}(\widetilde{\mu_n} - \mu)$$

$$= \frac{\mu\sqrt{T_n}}{\widetilde{\theta_n}}\left(\theta - \widetilde{\theta_n}\right) + \frac{1}{\sqrt{T_n}}\left(\int_0^{T_n} X_t\,dt - \Delta_n\sum_{i=1}^n X_{t_{i-1}}\right) + \frac{\left(\theta - \widetilde{\theta_n}\right)}{\widetilde{\theta_n}\sqrt{T_n}}\int_0^{T_n} X_t\,dt + \frac{B_{T_n}^H}{\widetilde{\theta_n}\sqrt{T_n}}$$

$:= A_n + B_n + C_n + D_n$.

Theorem 3.2 and the convergence (3.22) imply that $A_n \to 0$ in probability.

We can write $C_n = \frac{(\theta-\widetilde{\theta_n})}{\widetilde{\theta_n}\sqrt{T_n}}\int_0^{T_n} X_t\,dt = \frac{\sqrt{T_n}(\theta-\widetilde{\theta_n})}{\widetilde{\theta_n}}\left(\frac{1}{T_n}\int_0^{T_n} X_t\,dt\right)$.

Then, Theorem 3.2 and the convergence (3.22) imply that $\frac{\sqrt{T_n}(\theta-\widetilde{\theta_n})}{\widetilde{\theta_n}} \to 0$in probability. Moreover, using l'Hôpital rule,

$$\lim_{T_n\to\infty}\frac{1}{T_n}\int_0^{T_n} X_t\,dt = \lim_{T_n\to\infty} X_{T_n} = \lim_{T_n\to\infty}\left(\mu(1 - e^{-\theta T_n}) + \zeta_{T_n}\right) = \mu + \zeta_\infty.$$

Hence $C_n \to 0$in probability.

Recall that $E[(B_t^H - B_s^H)^2] = |t-s|^{2H}$; $t, s \ge 0$.

Then for $H \in \left]0, \frac{1}{2}\right[$, we have almost surely, as $T_n \to \infty$

$$\frac{B_{T_n}^H}{\sqrt{T_n}} \to 0, \quad \text{by Borel-Cantelli Lemma.}$$

Combining this with Theorem 3.2 we obtain that $D_n := \frac{B_{T_n}^H}{\widetilde{\theta_n}\sqrt{T_n}} \to 0$in probability.

$$B_n := \frac{1}{\sqrt{T_n}}\left(\int_0^{T_n} X_t\,dt - \Delta_n\sum_{i=1}^n X_{t_{i-1}}\right) = \frac{e^{\theta T_n}}{\sqrt{T_n}}\left(e^{-\theta T_n}\int_0^{T_n} X_t\,dt - e^{-\theta T_n}\Delta_n\sum_{i=1}^n X_{t_{i-1}}\right) \qquad (3.23)$$

By lemma 2.3, we have $e^{-\theta T_n}\int_0^{T_n} X_t\,dt \to \frac{1}{\theta}(\mu + \zeta_\infty)$ almost surely.

We also have

$$E\left[\left(e^{-\theta T_n}\Delta_n\sum_{i=1}^n X_{t_{i-1}}\right)^2\right] = \Delta_n^2 e^{-2\theta T_n}\sum_{i,j=1}^n E\left(X_{t_{i-1}}X_{t_{j-1}}\right) = \Delta_n^2 e^{-2\theta T_n}\sum_{i,j=1}^n e^{\theta t_{i-1}+\theta t_{j-1}}E\left(Z_{t_{i-1}}Z_{t_{j-1}}\right).$$

Then, by using the same arguments as in Lemma 3.2, we obtain

$\boldsymbol{E}\left[\left(e^{-\boldsymbol{\theta T}_n}\Delta_n\sum_{i=1}^{n}X_{t_{i-1}}\right)^2\right] \leq C(\mu,\theta,H)\Delta_n^2 e^{-2\theta T_n}\left(\frac{e^{\theta n\Delta_n}-1}{e^{\theta\Delta_n}-1}\right)^2 \leq C(\mu,\theta,H)\Delta_n^2 \to 0.$ (3.24)

Combining (2.10), (3.23), (3.24), and the fact that $\frac{e^{\theta T_n}}{\sqrt{T_n}} \to \infty$, we conclude that $B_n \to \infty$.

Consequently, the convergence (3.17) is proved. Thus the desired results are obtained. □

Theorem 3.2. Assume that $0 < H < 1$. Suppose that $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to 0$ for some α>0. Then as $n \to \infty$,
$\widetilde{\theta_n} \to \theta$almost surely. (3.25)

Proof. We can write

$$\widetilde{\theta_n} = \frac{\frac{1}{2}X_{T_n}^2 - \frac{X_{T_n}}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{\Delta_n\sum_{i=1}^{n}X_{t_{i-1}}^2 - \frac{\Delta_n}{n}\left(\sum_{i=1}^{n}X_{t_{i-1}}\right)^2}$$

$$= \frac{\frac{1}{2}e^{-2\theta T_n}X_{T_n}^2 - Z_{T_n}D_n}{e^{-2\theta T_n}S_n - \left(\sqrt{n\Delta_n}D_n\right)^2}.$$

Thus, according to (2.9) , (3.4) , (3.5) and (3.13), we can deduce that
$\widetilde{\theta_n} \to \theta$almost surely as $n \to \infty$. □

References:

[1] Alazemi, F., Alsenafi A., Es-Sebaiy, K. (2020). Parameter estimation for Gaussian mean-reverting Ornstein-Uhlenbeck processes of the second kind: non-ergodic case, Stochastics and Dynamics19, 2050011.
[2] Chronopoulou, A. and F.G. Viens (2012) Estimation and pricing under long-memory stochastic volatility. Annals of Finance 8, 379-403.
[3] Chronopoulou, A. and F.G. Viens (2012) Stochastic volatility and option pricing with long-memory in discrete and continuous time. Quantitative Finance 12, 635-649.
[4] Comte, F., Coutin, L. and Renault, E. (2012) Affine fractional stochastic volatility models. Annals of Finance 8, 337-378.
[5] Comte, F. and Renault, E. (1998). Long Memory in Continuous-time Stochastic Volatility Models. Mathematical Finance, 8(4):291-323.
[6] El Machkouri, M., Es-Sebaiy, K. and Ouknine, Y.(2016). Least squares estimator for non-ergodic Ornstein-Uhlenbeck processes driven by Gaussian processes. Journal of the Korean Statistical Society 45, 329-341.
[7] El Onsy, B., Es-Sebaiy, K. and Ndiaye, D. (2018). Parameter estimation for discretely observed non-ergodic fractional Ornstein-Uhlenbeck processes of the second kind. Brazilian Journal of Probability and Statistics, Vol. 32, No. 3, 545-558.
[8] El Onsy, B., Es-Sebaiy, K. and Viens, F. (2017). Parameter Estimation for a partially observed Ornstein-Uhlenbeck process with long-memory noise. Stochastics, Volume 89 (2017) - Issue 2, Pages: 431-468.
[9] Es-Sebaiy, K. and Es.Sebaiy, M. (2021) Estimating drift parameters in a non-ergodic Gaussian Vasicek-type model, Statistical Methods & Applications , Springer ; Societa Italiana di Statistica, ` vol. 30(2), pages 409-436, juin.
[10] Es-Sebaiy, K., Alazemi, F. and Al-Foraih, M. (2019). Least squares type estimation for discretely observed non-ergodic Gaussian Ornstein-Uhlenbeck processes. Acta Mathematica Scientia,39 , 989–1002.
[11] Hu, Y. and Nualart, D. (2010). Parameter estimation for fractional Ornstein-Uhlenbeck processes. Statist. Probab. Lett. 80, 1030-1038.
[12] Kloeden P, Neuenkirch A. The pathwise convergence of approximation schemes for stochastic differential equations. LMS J. Comp. Math., 2007, 10, 235-253.
[13] Nourdin, I. (2012). Selected aspects of fractional Brownian motion. Bocconi & Springer Series 4. Springer, Milan; Bocconi University Press, Milan.
[14] Vasicek O., 1977. An equilibrium characterization of the term structure. J. Finance Econ. 5(2): p. 177-188.
[15] Xiao, W. and Yu, J. (2018). Asymptotic theory for estimating drift parameters in the fractional Vasicek model. Econometric Theory, Page 1 of 34. doi:10.1017/S0266466618000051
[16] Young, L. C. (1936). An inequality of the Holder type connected with Stieltjes integration. Acta Math. ¨ 67, 251-282.