ISSN NO. 2320-5407

Journal homepage: http://www.journalijar.com

INTERNATIONAL JOURNAL OF ADVANCED RESEARCH

RESEARCH ARTICLE

COMPOSITE POLYTROPIC MODELS OF THE SUN.

***M. Sedek[1], H. El-Zafarani[1,3], Magdy Y. Amin [2,4] and Hayman Z. M.[2]**

1. National Research Institute of Astronomy and Geophysics, Helwan, Cairo, Egypt.
2. Astronomy Dept., Faculty of Science, Cairo University, Cairo, Egypt.
3. Physics Dept., Faculty of Sciences and Arts, Al-Mandaq, Al Baha University, Kingdom of Saudi Arabia.
4. Physics Dept., College of Sciences and Humanities, Hawtat Sudair, Majmaah University, Kingdom of Saudi Arabia.

Manuscript Info

Manuscript History:

Received: 14 January 2016
Final Accepted: 26 February 2016
Published Online: March 2016

Key words:
Breeding season,
Buffaloe, Post- partum anoestrus.

Corresponding Author

M. Sedek.

Abstract

One of the most important contributions which stellar structure has made for equilibrium configuration is the theory of self gravitating polytrope. In this paper we use the Standard Solar Model SSM of Bahcall, Serenelli, Basu(2005) as a basic to developed a solar models in hydrostatic equilibrium expressing the physical structure of a sequence of two zones polytropes associated to the convective and radiative regions, and three zones polytropes associated to the nuclear, the radiative and the convective regions of the solar interior. We show that a composite three zones polytrope is a good representation for SSM.

Introduction:-

Polytropes are useful as they provide simple solutions for the internal structure of a star that can be tabulated and used for estimates of various quantities. They are much simpler to manipulate than the full rigorous solutions of all the equations of stellar structure. The equation of a polytropic state is a power law relationship between pressure and density:

$$P = K\,\rho^{\Upsilon} = K\,\rho^{1+\frac{1}{n}} \tag{1}$$

where Υ is the adiabatic index (a parameter characterizing the behavior of the specific heat of a gas), K is a polytropic constant and n is called the polytropic index.

As suggested first by Emden in 1908 a simplified stellar model considers a star as an ideal gas self-gravitating in hydrostatic equilibrium. The equations governing the hydrostatic equilibrium are (Chandrasekhar, S., 1939)

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2} \tag{2}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho \tag{3}$$

$M(r)$ is the enclosed mass under the radius r. Eliminating $M(r)$ between these two equations and use equation (1), we get

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \tag{4}$$

If one is going to investigate the general solution-set of any equation, it is usually a good idea to express the equation in a dimensionless form. This can be done to equation (4) by transformation to the so-called Emden

$$\rho = \rho_0 \theta^n \qquad , \qquad r = \alpha\xi \tag{5}$$

where,

$$\alpha = \left(\frac{(n+1)K\rho_0^{\frac{1}{n}-1}}{4\pi G}\right)^{1/2} \tag{6}$$

We then get the more familiar form of the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{1}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{7}$$

The Lane-Emden equation is basis of all polytropic models and has analytical solutions for $n = 0, 1, and\ 5$. For all other polytrope indices n, a numerical solution to the Lane-Emden equation must be calculated. We used 4th order Runge-Kutta method to integrate it. The mass-radius relation can be written as (Horedt, G.P., 2004)

$$Mr^{3-n/n-1} = 4\pi\left[\frac{K(n+1)}{4\pi G}\right]^{n/n-1} \xi^{(n+1)/(n-1)} \left(\mp\frac{d\theta}{d\xi}\right) \tag{8}$$

Also, the temperature T and the gravitational acceleration F can be written as (El-Zafarani, H., 1999)

$$T = (\mu\, P_0/\mathcal{R}\,\rho_0)\,\theta \tag{9}$$

$$F = -[\pm 4\pi G\,(n+1)K]^{1/2}\rho_0^{(n+1)/2n}\,(\mp\, d\theta/d\xi) \tag{10}$$

The following two functions u , v introduced by Milne (1930, 1932) play important role in fitting up solutions at the interface of the composite stellar models: (Menzel et al., 1963)

$$u(\xi) = \mp\xi\,\frac{\theta^n}{\theta'} \qquad ; \qquad v(\xi) = \mp\xi\,\frac{\theta'}{\theta} \tag{11}$$

Polytropic Models of the Sun:-

Our knowledge of the sun is derived from sophisticated standard solar model (SSM) (Bahcall et al., 2005). The solar matter is at present approximately 75% hydrogen, 23% helium and 2% metals by mass fraction. Throughout the solar interior, μ is approximately 0.59, except at the surface ($\mu = 0.613$), and in the core ($\mu = 0.846$).

When we have used the pressure (P) and the density (ρ) data of Standard Solar Model SSM (2005) to plot a relation between the polytropic exponent ($\gamma = d\ lnP/d\ ln\rho$) and distance from center of sun in unit of solar radius (R/R_{sun}), we get a graph that consist of three regions (see Fig.1). These regions are corresponding to the core, the radiative zone and the convective zone.

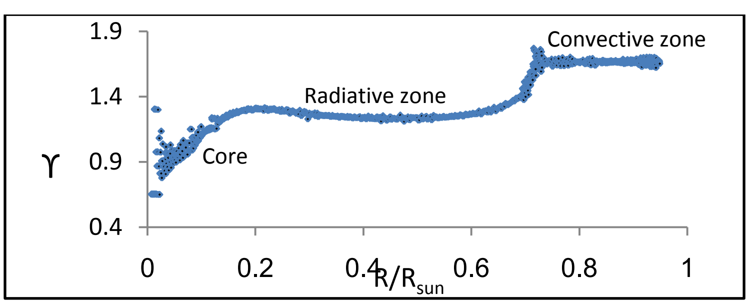

Fig.1: Polytropic exponent (ϒ) along the interior of sun.

Two zones solar model:-

We now will construct a model of sun with two polytropic regions, representing the convective and the radiative zones. From Fig.1, We take the polytropic index $n_1 = 1.5\,(\gamma_1 = 1.667)$ for the convective zone, and take the polytropic index $n_2 = 3.85\,(\gamma_2 = 1.260)$ for the radiative zone. To avoid confusion, we use the dimensionless variables ξ , θ for the convective zone and the variables $\eta\,,\phi$ for the radiative zone. Then, we have the equations of state for the convective and the radiative zone

$$P_1 = K_1\,{\rho_1}^{1.67} \qquad \text{where} \qquad \rho_1 = \rho_{1o}\theta(\xi)^{1.67} \tag{12}$$

$$P_2 = K_2\,{\rho_2}^{1.26} \qquad \text{where} \qquad \rho_2 = \rho_{2o}\phi(\eta)^{1.26} \tag{13}$$

The core region will correspond to an E-solution of Lane-Emden equation (7) with polytropic index $n_2 = 3.85$, so the Lane-Emden equation for the core is

$$(1/\eta^2)\,1/d\eta\,(\eta^2\ d\phi/d\eta) = -\phi^{3.85} \tag{14}$$

We integrate equation (14) from the center to surface by using 4$^{\text{th}}$-order Runge-Kutta method, then the values of η_i , $\phi(\tau_i)$, $\phi'(\tau_i)$ are obtained. Translate these values into the $u\,,V$ variables and draw it in (u,V) plane as shown in Fig.2 .

The Lane-Emden function for convective zone is

$$(1/\xi^2)\,1/d\xi\,(\xi^2\ d\theta/d\xi) = -\theta^{1.5} \tag{15}$$

The M-solutions of the Lane-Emden equation (15) are of particular interest because these are the ones which intersect the polytrope of the radiative zone. These may be generated beginning at ξ_1 with an arbitrary starting slope $\theta'(\xi)$ less negative than $\theta'(\xi_1)$ [ξ_1 , $\theta'(\xi_1)$ are the values of ξ and θ' at first zero of Lane-Emden function θ] and integrating inward. Three such solutions are shown in Fig.2.

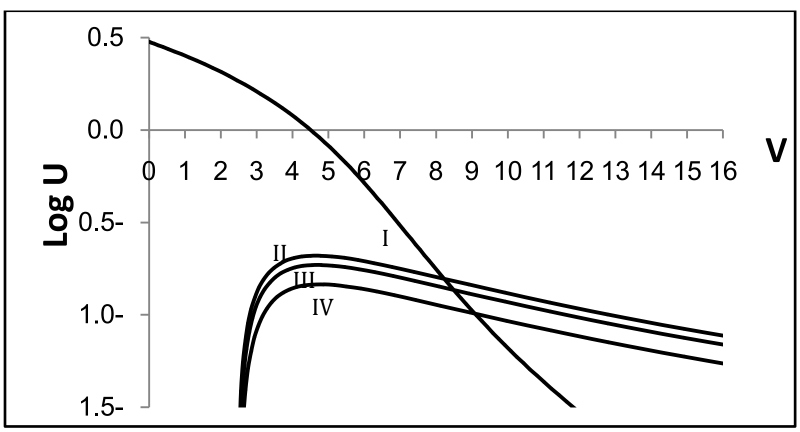

Fig.2: The curve I is an E-solutions of Lane-Emden equation of polytropic index $n_2 = 3.85$. The curves II, III and IV are M-solutions for the convective zone with starting slope $\theta'(\xi_1) = -0.004\,, -0.00325\ and\ -0.002$ respectively.

First, let us start with the convective zone. We choose the solution with starting slope $\theta'(\xi_1)\ = -0.0032$ because as we will see blow, this gives the depth of convective zone at $R = 0.71\,R_{SUN}$ and gives a good approximation to SSM (Bahcall, J., Serenelli, A., Basu, S.: 2005(BS2005)).

Using the values of mass M and radius R of the sun in equations (8) and (5), we get the values of the pressure constant K_1 and the constant of homology ρ_{1o} for the convective zone.

$$K_1 = 9.892 \times 10^8 \quad , \quad \rho_{1o} = 5.351 \times 10^5 \tag{16}$$

We determine the intersection point $[(u_i, V_i) = (0.1373511\ ,\ 8.4788239)]$ between the M-solution of the convective zone and E-solution of the radiative zone in the (u, V) plane [Fig.2] and the corresponding values of $\xi_i = 2.5927674$ and $\theta_i = 0.0047453$ by cubic-spline interpolation. The density ρ_{1i} and the pressure P_{1i} at this point are

$$\rho_{1i} = \rho_{1o}\ \theta_i^{1.5} = 1.749 \times 10^2 \tag{17}$$

$$P_{1i} = K_1\ {\rho_i}^{1+\frac{1}{1.5}} = 5.412 \times 10^{12} \tag{18}$$

Secondly, the radiative zone: The intersection point (u_i, V_i) between the radiative zone and the convective zone is corresponding to $\eta_i = 5.6345837$ and $\phi_i = 0.1801321$. At this point, the density and the pressure of the radiative zone and of the convective zone are equal in order to satisfy the continuity, so we have

$$\rho_{2i} = \rho_{1i} = \rho_{2o}\ \phi_{ie}^{n2} = 1.749 \times 10^2 \tag{19}$$

$$P_{2ie} = P_{1i} = K_2\ {\rho_{2ie}}^{1+\frac{1}{n2}} = 5.412 \times 10^{12} \tag{20}$$

Then the values of the pressure constant K_2 and the homology constant ρ_{2o} are

$$K_2 = 8.090 \times\ 10^9 \tag{21}$$

$$\rho_{2o} = 1.285 \times 10^5 \tag{22}$$

By knowing $\theta(\xi), \phi(\eta)$, K_1 , ρ_{1o}, K_2 and ρ_{2o} . The whole structure of the two zone polytrope is now completely determined. The numerical values of the physical quantities of our solar model are calculated and it is seen in table 1, where the first two columns contain radius and mass relative to radius and mass of the sun respectively. The four columns contain density (ρ), pressure (P), temperature (T) and gravitational acceleration (F). All physical quantities are expressed in SI unit.

Table 1: Numerical values of two-zone solar model.

R/R_{sun}	M/M_{sun}	ρ	P	T	F
0.05	1.076E-02	1.164E+05	1.948E+16	1.197E+07	-1.178E+03
0.10	7.292E-02	8.810E+04	1.372E+16	1.113E+07	-1.996E+03
0.15	1.926E-01	5.830E+04	8.156E+15	1.000E+07	-2.343E+03
0.20	3.410E-01	3.546E+04	4.360E+15	8.789E+06	-2.333E+03
0.25	4.869E-01	2.066E+04	2.207E+15	7.638E+06	-2.132E+03
0.30	6.131E-01	1.185E+04	1.096E+15	6.612E+06	-1.864E+03
0.35	7.143E-01	6.806E+03	5.450E+14	5.724E+06	-1.596E+03
0.40	7.921E-01	3.946E+03	2.743E+14	4.969E+06	-1.355E+03
0.45	8.505E-01	2.318E+03	1.404E+14	4.328E+06	-1.149E+03
0.50	8.936E-01	1.381E+03	7.311E+13	3.783E+06	-9.783E+02
0.55	9.253E-01	8.341E+02	3.872E+13	3.318E+06	-8.372E+02
0.60	9.483E-01	5.095E+02	2.081E+13	2.920E+06	-7.210E+02
0.65	9.650E-01	3.141E+02	1.132E+13	2.575E+06	-6.252E+02
0.70	9.771E-01	1.950E+02	6.208E+12	2.275E+06	-5.458E+02
0.75	9.837E-01	1.291E+02	3.263E+12	1.806E+06	-4.786E+02
0.80	9.905E-01	8.417E+01	1.599E+12	1.358E+06	-4.236E+02
0.85	9.953E-01	5.004E+01	6.722E+11	9.602E+05	-3.770E+02

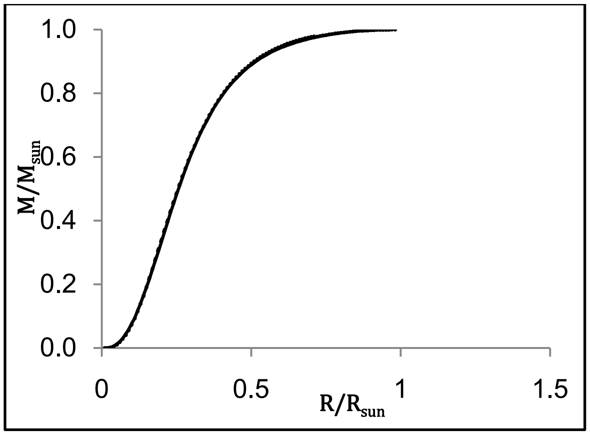

Fig. 3: the mass-radius relation for our two zone solar model (dashed) and for SSM (solid line)

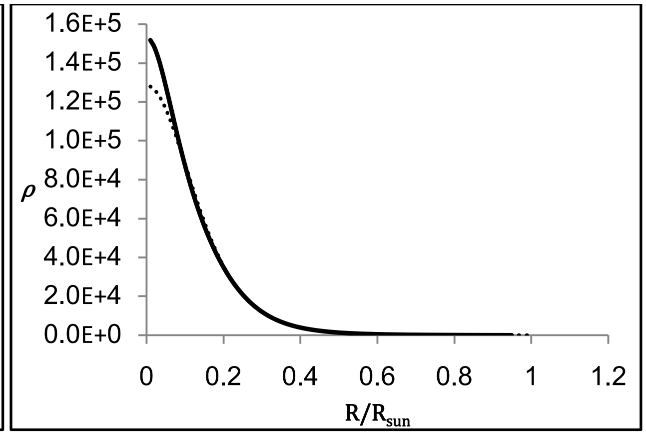

Fig. 4: Shows the density distribution for our two-zone solar model (dashed) and for SSM (solid line).

Three-zones solar model:-

This model is based on that the solar interior, from the inside out, is made up of the core, radiative zone and the convective zone. From Fig.1, the average value of the polytropic index, along the convective zone is $n_1 = 1.5$ (i.e. $\Upsilon_1 = 1.667$), along the radiative zone is $n_2 = 3.7878$ (i.e. $\Upsilon_2 = 1.264$) and along the core is $n_3 = 20$ (i.e. $\Upsilon_3 = 1.05$).We use ξ, θ as the variables in the Lane-Emden equation for the convective zone; η, ϕ as the variables for the radiative zone and τ, ψ as the variables for the nuclear zone. Then, the equations of state for three zones are

$$P_1 = K_1 \, \rho_1{}^{1.67} \qquad \text{where} \qquad \rho_1 = \rho_{1o} \theta(\xi)^{1.67} \tag{23}$$

$$P_2 = K_2 \, \rho_2{}^{1.26} \qquad \text{where} \qquad \rho_2 = \rho_{2o} \phi(\eta)^{1.26} \tag{24}$$

$$P_3 = K_3 \, \rho_3{}^{1.05} \qquad \text{where} \qquad \rho_3 = \rho_{3o} \psi(\tau)^{1.05} \tag{25}$$

The core region will correspond to an E-solution of Lane-Emden equation, so Lane-Emden equation for the core is

$$(1/\tau^2)\, 1/d\tau \, (\tau^2 \; d\psi/d\tau) = -\psi^{20} \tag{26}$$

By integrating equation (26) outward using 4th-order Runge-Kutta method, we get the values of τ_i , $\psi(\tau_i)$, $\psi'(\tau_i)$ then translated these values into the (u, V) variables and draw it in (u, V) plane as shown in Fig.5 . Note that the polytrope of index equal to 20 has infinite radius (i.e. $\tau_1 = \infty$ where τ_1 is first zero of Lane-Emden function ψ).

The Lane-Emden function for convective zone is

$$(1/\xi^2)\, 1/d\xi \, (\xi^2 \; d\theta/d\xi) = -\theta^{1.5} \tag{27}$$

and the Lane-Emden function for radiative zone is

$$(1/\eta^2)\, 1/d\eta \, (\eta^2 \; d\phi/\mathrm{d}\eta) = -\phi^{3.7878} \tag{28}$$

The convective zone will represent by M-solutions of Lane-Emden equation (27). These may be generated beginning at ξ_1 with an arbitrary starting slope $\theta'(\xi)$ less negative than $\theta'(\xi_1)$ [ξ_1 , $\theta'(\xi_1)$ are the values of ξ and θ' at first zero of Lane-Emden function θ] and integrating inward. Three such solutions are shown in Fig.5. In the same way, we can get M-solutions that represent the radiative zone. Three of such solutions with staring slope $\phi'(\eta)$ less negative than $\phi'(\eta_1)$ [η_1 , $\phi'(\eta_1)$ are the values of η , ϕ' at first zero of Lane-Emden function ϕ] are shown in Fig.5.

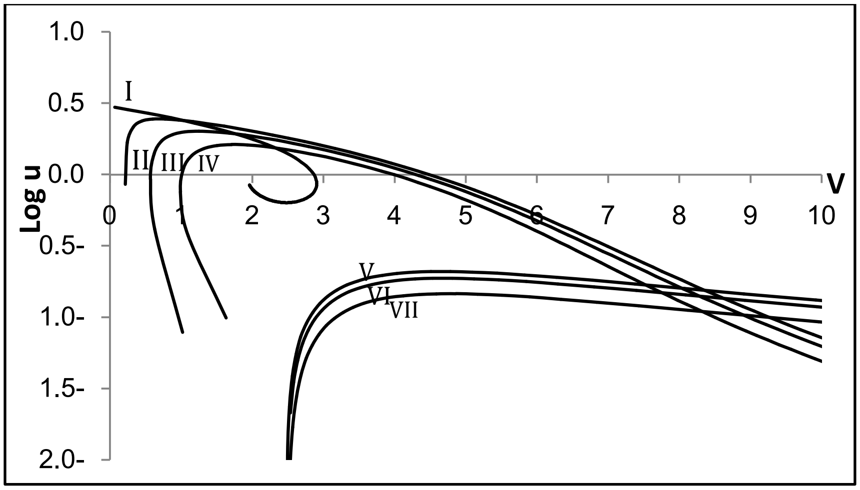

Fig.5:- The curve I is an E-solution of the core. The curves II, III and IV are M-solutions for the radiative zone with starting slope $\phi'(\eta_1) = -0.012325, -0.0117$ *and* -0.0107 respectively. The curves V, VI and VII are M-solutions for the convective zone with starting slope $\theta'(\xi_1) = -0.004, -0.00325$ *and* -0.002 respectively.

Let us choose the solution with stating slope $\theta'(\xi_1) = -0.00325$ for the convective zone and the solution with stating slope $\phi'(\eta_1) = -0.012325$ for the radiative zone because these give the depth of convective zone at $R = 0.714\ R_{sun}$ and give a good approximation to SSM.

First, let us start with the convective zone, since the radius and mass of sun are known, we can use equations (5) and (8) to get the values of K_1 and ρ_{1o}.

$$K_1 = 9.841 \times 10^8 \quad , \quad \rho_{1o} = 5.269 \times 10^5 \tag{29}$$

At the intersection point [$(u_i, V_i)_e = (0.135577\ ,\ 8.622205)$] between the M-solution of the convective zone and of the radiative zone in the (u, V) plane [Fig.5] which is corresponding to $\xi_i = 2.609952$ and $\theta_i = 0.004712$. This point and the corresponding values of ξ_i , θ_i are determined by cubic-spline interpolation. From equations (23), the density ρ_{1i} and the pressure P_{1i} at this point are

$$\rho_{1i} = \rho_{1o}\ \theta_i^{1.5} = 1.704 \times 10^2 \tag{30}$$

$$P_{1i} = K_1\ \rho_i^{\ 1+\frac{1}{1.5}} = 5.154 \times 10^{12} \tag{31}$$

Secondly, the radiative zone: The intersection point $(u_i, V_i)_e$ between radiative zone and convective zone is corresponding to $\eta_{ie} = 5.507149$ and $\phi_{ie} = 0.177308$. At this point, the density and the pressure of radiative zone and of convective zone are equal in order to satisfy the continuity. So from equations (24), (30) and (31), we get

$$\rho_{2ie} = \rho_{1i} = \rho_{2o}\ \phi_{ie}^{n_2} = 1.704 \times 10^2 \tag{32}$$

$$P_{2ie} = P_{1i} = K_2\ \rho_{2ie}^{\ 1+\frac{1}{n2}} = 5.154 \times 10^{12} \tag{33}$$

From equations (32) and (33), we obtain the values of the pressure constant K_2 and the homology constant ρ_{2o} .

$$K_2 = 7.791 \times\ 10^9 \quad , \quad \rho_{2o} = 1.194 \times 10^5 \tag{34}$$

At the intersection point [$(u_i, V_i)_c = (2.383569\ ,\ 1.048965)$] between the M-solution of the radiative zone and E-solution of the core in the (u, V) plane [Fig.5] which is corresponding to $\eta_{ic} = 0.828439$ and $\phi_{ic} = 0.906953$. The density ρ_{2ic} and the pressure P_{2ic} are obtained by

$$\rho_{2ic} = \rho_{2\mathrm{o}}\ \phi_{\mathrm{ic}}^{\mathrm{n}2} = 8.25 \times 10^4 \tag{35}$$

$$P_{2ic} = K_2\, \rho_{2ic}^{1+\frac{1}{n2}} = 1.276 \times 10^{16} \tag{36}$$

Finally, the core: The intersection point $(u_i, V_i)_c$ between the M-solution of the radiative zone and E-solution of the core is corresponding to $\tau_i = 0.455335$ and $\psi_i = 0.906953$. At this point, the density and the pressure of core and of radiative zone are equal in order to satisfy the continuity. Then,

$$\rho_{3i} = \rho_{2ic} = \rho_{3o}\psi_i^{n_3} = 8.25 \times 10^4 \tag{37}$$

$$P_{3i} = P_{2ic} = K_3\, \rho_{3i}^{1+\frac{1}{n3}} = 1.276 \times 10^{16} \tag{38}$$

From equations (37) and (38), we get the values of the pressure constant K_3 and the homology constant ρ_{3o} (ρ_{3o} is central density of configuration).

$$K_3 = 8.785 \times 10^{10} \quad , \quad \rho_{3o} = 1.483 \times 10^5 \tag{39}$$

We see that our model of the sun yields central density $\rho_c = 1.483 \times 10^5$ Kg m^{-3} which is very close to that of standard solar model $\rho_c = 1.529 \times 10^5$ Kg m^{-3}.

Now, the values of $\theta(\xi)$, $\theta^{'}(\xi)$, K_1 , ρ_{1o} for the convective zone, $\phi(\eta)$, $\phi^{'}(\eta)$, K_2 , ρ_{2o} for the radiative zone, and $\psi(\tau)$, $\psi^{'}(\tau)$, K_3 , ρ_{3o} for the core are known. So we can compute the physical march from the center to the surface as shown in Table 2.

Table 2: Numerical values of Three-zones solar model.

R/R_{sun}	M/M_{sun}	ρ	P	T	F
0.05	0.0121	1.283E+05	2.030E+16	1.609E+07	-1.323E+03
0.10	0.0770	8.828E+04	1.370E+16	1.579E+07	-2.107E+03
0.15	0.1950	5.731E+04	8.054E+15	1.005E+07	-2.372E+03
0.20	0.3408	3.490E+04	4.303E+15	8.813E+06	-2.332E+03
0.25	0.4848	2.042E+04	2.186E+15	7.650E+06	-2.123E+03
0.30	0.6098	1.178E+04	1.090E+15	6.616E+06	-1.854E+03
0.35	0.7106	6.797E+03	5.441E+14	5.722E+06	-1.588E+03
0.40	0.7885	3.956E+03	2.745E+14	4.960E+06	-1.349E+03
0.45	0.8471	2.331E+03	1.406E+14	4.313E+06	-1.145E+03
0.50	0.8905	1.391E+03	7.324E+13	3.764E+06	-9.749E+02
0.55	0.9224	8.401E+02	3.872E+13	3.295E+06	-8.346E+02
0.60	0.9456	5.126E+02	2.074E+13	2.892E+06	-7.189E+02
0.65	0.9624	3.154E+02	1.122E+13	2.544E+06	-6.234E+02
0.70	0.9745	1.951E+02	6.116E+12	2.241E+06	-5.443E+02
0.75	0.9836	1.301E+02	3.288E+12	1.806E+06	-4.786E+02
0.80	0.9904	8.482E+01	1.611E+12	1.358E+06	-4.235E+02
0.85	0.9952	5.043E+01	6.774E+11	9.602E+05	-3.770E+02

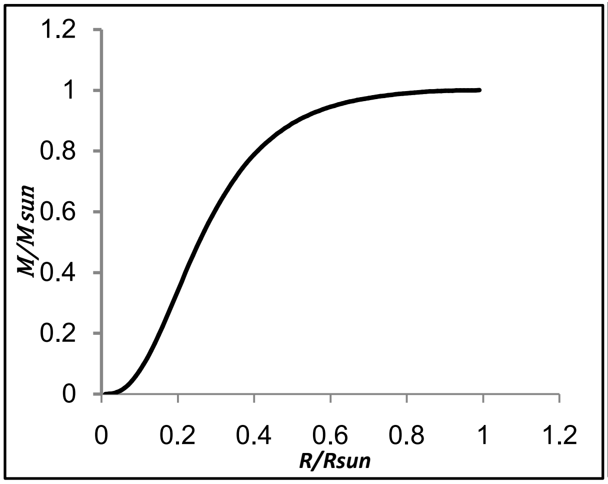

Fig. 6: the mass-radius relation for our three-zone solar model (dashed) and for SSM (solid line).

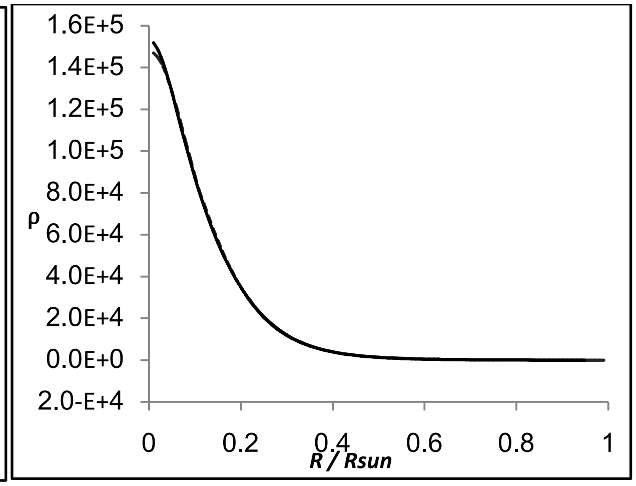

Fig. 7: Shows the density distribution for our three-zone solar model (dashed) and for SSM (solid line).

References:-

1. **Chandrasekhar, S. (1939).** An Introduction to the Study of Stellar Structure, Chicago University press, Chicago. (Dover Edition, 1958)
2. **Bahcall, J., Serenelli, A., Basu, S. (2005).** New Solar Opacities, Abundances, Helioseismology and Neutrino Fluxes, ApJ, 621, L85.
3. **El-Zafarani**, **H. (1999).** Physical Characteristic of N-dimensional Radially Symmetric Polytropes, MSc thesis, Astronomy Dep., Cairo University, Egypt.
4. **Horedt, G. P. (2004)**. Polytropes: Applications in astrophysics and related fields, Kluwer, Dordrecht.
5. **Menzel, D., Bhatnagar, P. L. and sen, H. K. (1963).** Stellar Interiors, Chapman & Hall, London.