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RESEARCH ARTICLE

NEARLY PRIMARY SUBMODULES

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Abstract

Let R be a commutative ring with unity and let N be a submodule of a nonzero left R-module M, N is called primary if whenever $rx \in N$, $r \in R, x \in M$, implies that either $x \in N$ or $r \in \sqrt{[N:M]}$. In this paper we say that N is nearly primary, if whenever $rx \in N$, $r \in R, x \in M$,, implies that either $x \in N + J(M)$ or $r \in \sqrt{[N + J(M): M]}$, (in short N. primary),where $J(M)$ is the Jacobson radical of M. We give many results of this type of submodules.

INTRODUCTION

Let R be a commutative ring with identity and let M be an R-module. A proper ideal I of a commutative ring R with identity is called primary ideal if for each $a, b \in R$ such that $a.b \in I$ and $a \notin I$, then $\exists\, k \in Z^+$ such that $b^k \in I$ [1]. A proper submodule N of an R-module M is called primary submodule if whenever $r \in R$, $m \in M$ such that $rm \in N$, then either $m \in N$ or $r^k \in [N:M]$, for some $k \in Z^+$ [2]. Primary submodule which is a generalization to primary ideals is studied by many mathematicians. A proper ideal I in a ring R is called prime ideal if $a.b \in I$ implies that either $a \in I$ or $b \in I$. [1]. A proper submodule N of an R-module M is called prime submodule if $N \neq M$ and whenever $r\,m \in N$, $r \in R, m \in M$, implies that either $m \in N$ or $r \in [N:M]$ [3]. A proper submodule N of an R-module M is called semiprime submodule if whenever $r \in R$, $m \in M$, $k \in Z^+$ such that $r^k m \in M$, then $r\,m \in N$. [4], equivalently, if N is a proper submodule N of an R-module M, then N is semiprime submodule of M if and only if whenever $r^2 m \in N$, where $r \in R, m \in M$, then $r\,m \in N$. We know that every prime submodule is primary submodule, but the converse is not true in general for examble $4Z$ is a primary submodule of Z as $Z -$ module, but not a prime submodule of Z as $Z -$ module. N be a proper submodule of an $R -$module M, then N is a primary and semiprime submodule of M iff it is a prime submodule of M[5]. N be a proper submodule of an $R -$module M, then the following conditions are hold: N is a primary submodule of M, $[N:K]$ is a primary ideal of R, for each $N \subseteq K$, $[N:\langle x\rangle]$ is a primary ideal of R, for each $x \notin N$, where $[N:K] = \{r \in R: rK \subseteq N\}$. In this paper we study the concept of nearly primary submodules as a generalization of the concept of primary submodules, where a proper submodule N of an $R -$module M is called nearly primary if whenever $r \in R$, $m \in M$ such that $rm \in N$, then either $m \in N + J(M)$ or $r^k \in [N + J(M): M]$, for some $k \in Z^+$. In this work we give new results (up to our knowledge) about this concepts, illustrate that by some remarks and examples. We see that it is necessary to start by the formula of the definition of primary submodules and put some simple remarks about primary submodules which were mentioned in [9], [11].

preliminaries

Let R be a commutative ring with identity and let M be a nonzero unite left R-module, N is called primary submodule if whenever $rx \in N$, $r \in R, x \in M$, implies either $x \in N$ or $r \in \sqrt{[N:M]}$ [2].

A submodule A of an R-module M is called small (for short $A \ll M$), if whenever $A + B = M$, for some submodule B of M implies $B = M$ [7].A proper submodule N of an R-module M is called maximal if whenever $N \subsetneq K \subseteq M$ implies $K = M$ [7] . We know that Jacobson radical of M (for short $J(M)$) is denoted by the intersection of all maximal submodule of M and denoted by the sum of all small submodule of M[7]. If $\phi: M \longrightarrow M'$ is an R-homomorphism, then $\phi(J(M)) \subseteq J(M')$, If $\phi: M \longrightarrow M'$ is an R- epimorphism and $\ker\emptyset \ll M$, then $\phi(J(M)) = J(M')$, and $J(R).M \subseteq J(M)$, where R is a ring, if (M is projective module[4] or R is good ring[7] or R is local ring), then $J(R).M = J(M)$.[7]. An R-module M is said to be multiplication module if each submodule N of M, then there exists an ideal I of R such that $IM = N$, in fact M is called multiplication module if $[N:M]M = N, N \subseteq M$, where $[N:M] = \{r \in R : rM \subseteq N\}$.[8]

1. primary Submodules

Recall that a proper ideal I of a commutative ring R with identity is called primary ideal if for each $a, b \in R$ such that $a.b \in I$ and $a \notin I$, then $\exists\ k \in Z^+$ such that $b^k \in I$ [1].

Recall that a proper submodule N of an R-module M is called primary submodule if whenever $r \in R$, $m \in M$ such that $rm \in N$, then either $m \in N$ or $r^k \in [N:M]$, for some $k \in Z^+$ [2].

Recall that a proper ideal I of a ring R is called prime ideal if $a.b \in I$ implies that either $a \in I$ or $b \in I$. [1]

Remarks and examples (2.1) :

1)Every prime submodule of an R –module M is a primary submodule of M, but the convers is not true in general for example 4Z is a primary submodule of Z as $Z -$ module, but not a prime submodule of Z as $Z -$ module.
2)6Z is neither a primary nor a prime submodule of Z as $Z -$ module.
3)In $Z \oplus Z$ over Z the submodule $N = 2Z \oplus \langle 0 \rangle$ is not a primary submodule beccause $(6,0) = 2(3,0) \in N$, neither $(3,0) \in N$ nor $2^k \in [N:M] = 0$, for some $k \in Z^+$, but it is a semiprime submodule .
4)Every maximal ideal is a primary ideal, but the converse is not true in general, for example 8Z is a primary submodule of Z as $Z -$ module, but it is not maximal .
5)Every prime ideal is a primary ideal , but the converse is not true in general, for example 8Z is a primary submodule of Z as $Z -$ module, but it is not prime .

Remark(2.2) :
If N is a prime submodule of an R-module M , then $[N:M]$ is a prime ideal of R, but the converse is not true in general, for example let $N = \langle 2,0 \rangle$ be a submodule of a module $Z \oplus Z$ over Z it is clear that $[N:M] = 0$ is a prime of Z, but is not a prime submodule of $Z \oplus Z$. [9]

For the converse we have the following :

Proposition (2.3) :
Let N be a primary submodule of an $R -$module M, then N is a prime submodule of M iff $[N:M]$ is a prime ideal of R, [9]

Proposition (2.4) :
If K is a primary submodule of an R –module M and contain a prime submodule N of M such that $[N:M] = \sqrt{[K:M]}$, then K is a prime submodule of M, [9]

Proposition (2.5):
Let N be a proper submodule of an $R -$module M, then N is a primary and semiprime submodule of M iff it is a prime submodule of M, [5]
Proposition (2.6): [10]
Let N be a proper submodule of an $R -$module M, then the following conditions are hold:

1) N is a primary submodule of M.
2) $[N:K]$ is a primary ideal of R, for each $N \subseteq K$.
3) $[N:\langle x\rangle]$ is a primary ideal of R, for each $x \notin N$.

Corollary (3.7):[11]
Let N be a primary submodule of an $R-$ module, then $[N:M]$ is a primary ideal of R and hence $\sqrt{[N:M]}$ is a prime ideal.
The converse of the corollary is not true in general, for example let $M = Z \oplus Z$ as $Z-$ module and consider the submodule $N = 2Z \oplus \langle 0\rangle$ of M, then $[N:M] = [2Z \oplus \langle 0\rangle : Z \oplus Z] = 0$, which is a primary ideal of Z , but N is not a primary submodule of M.

Proposition (2.8):[10]
Let N be a proper submodule of an $R-$ module M such that $[K:M] \nsubseteq [N:M]$, for each submodule K of M, then N is a primary submodule iff $[N:M]$ is a primary ideal of R.

Corollary (2.9):[10]
Let N be a proper submodule of a multiplication $R-$ module M, then N is a primary submodule iff $[N:M]$ is a primary ideal of R .

Corollary (2.10):[10]
Let N be a proper submodule of a cyclic $R-$ module M , then N is a primary submodule iff $[N:M]$ is a primary ideal of R .

Remark (2.11):[10]
The intersection of any two primary submodules of an $R-$ module M need not be a primary submodules of M, for example Z_6 as $Z-$ module has two primary submodules $N_1 = \langle\bar{2}\rangle$ and $N_2 = \langle\bar{3}\rangle$, but $N_1 \cap N_2 = \langle\bar{0}\rangle$ is not primary submodules of Z_6 .

Proposition (2.12):[10]
Let N and K be any two submodules of an $R-$ module M such that N is a primary submodule of M and K is not contain in N, then $N \cap K$ is a primary submodule of K

Proposition (2.13):[10]
Let M_1 and M_2 be two $R-$ modules and let $M = M_1 \oplus M_2$.If $N = N_1 \oplus N_2$ is a primary submodules of, then $N_1$and N_2 are primary submodules of M_1 and M_2 respectively.

Now, we introduce other results on primary submodules.

Compare the following with Proposition (1.3.8) in [12].

Proposition (2. 14):
If $\phi: M \longrightarrow M'$ is an $R-$isomorphism and N is a primary submodule of M, then $\phi(N)$ is a primary submodule of M'; where M and M' are two $R-$modules.

Proof: Let $r \in R, m' \in M'$ such that $rm' \in \phi(N)$, we have to show either $m' \in \phi(N)$ or $r^k \in [\phi(N):M']$, for some $k \in Z^+$,Suppose $r^k \notin [\phi(N):M']$, $m' \in M'$ and ϕ is onto , then $\exists\, m \in M$ such that $\phi(m) = m'$. $rm' = r\,\phi(m) = \phi(rm) \in \phi(N)$, $\exists\, y \in N$ such that $\phi(y) = \phi(rm)$, then $y - rm \in Ker\phi = \{0\}$, then $rm = y$, then $rm \in N$. But N is a primary submodule of M, then either $m \in N$ or $r^k \in [N:M]$. If $m \in N$, then $m' = \phi(m) \in \phi(N)$ or $r^k \in [N:M]$, then $r^k M \subseteq N$, thus $r^k \in N$, then $\phi(r^k m) \in \phi(N)$, then $r^k\phi(m) \in \phi(N)$, then $r^k m' \in \phi(N)$, then $r^k M' \subseteq \phi(N)$, then $r^k \in [\phi(N):M']$, which implise that $\phi(N)$ is a primary submodule of M'.

Proposition (2.15):

Let $\phi: M \longrightarrow M'$ be an R −homomorphism. If N is a primary submodule of M' such that $\phi(M) \nsubseteq N$, then $\phi^{-1}(N)$ is a primary submodule of M where M and M′ are two R −modules.

Proof: To show that $\phi^{-1}(N)$ is a proper submodule of. Suppose that $\phi^{-1}(N) = M$, let $m \in M$, then $m \in \phi^{-1}(N)$, then $\phi(m) \in N$ and this is a contradiction since $\phi(M) \nsubseteq N$.
Now Let $r \in R, m \in M$ such that $rm \in \phi^{-1}(N)$ and suppose that $m \notin \phi^{-1}(N)$, then $r\,\phi(m) = \phi(rm) \in N$, but N is a primary submodule of M' and $\phi(m) \notin N$, there for $r^k \in [N: M']$, this implies that $r^k M' \subseteq N$, since $\phi(M) \subseteq M'$, then $r^k\phi(M) = \phi(r^k M) \subseteq r^k M' \subseteq N$, then $r^k M \subseteq \phi^{-1}(N)$, then $r^k \in [\phi^{-1}(N): M\,]$. Which implise that $\phi^{-1}(N)$ is a primary submodule of M.

Recall that if N is a prime submodule of an R −module M, then $E_M(N) \subseteq N$. For primary submodules we have the following:

Proposition (2.16):
If N is a primary submodule of an R −module M, then $E_M(N) \nsubseteq N$.

Proof: Let $x = rm \in E_M(N)$, where $r \in R, m \in M, \exists\, n \in Z^+$ such that $r^k m \in N$ *(i.e.)* $r^k \in [N: M']$. N is a primary submodule of m , then $m \notin N$. Which implise that $E_M(N) \nsubseteq N$.

We introduce the following:

3. **Nearly primary Submodules.**

Definetion(3.1):

A proper submodule N of an R-module M is called nearly primary submodule if whenever $r \in R$, $m \in M$ such that $rm \in N$, then either $m \in N + J(M)$ or $r^k \in [N + J(M): M]$, for some $k \in Z^+$. Where J(M) is the jacobson radical of M.

Remarks and examples (3.2)**:**

1) Every primary submodule N of an R − module M is a nearly Primary submodule of M .
Proof: Let N be a proper submodule of an R− module M, we want to show that either $m \in N + J(M)$ or $r^k \in [N + J(M): M]$, for some $k \in Z^+$.
Let $r \in R$, $m \in M$ such that $rm \in N$. Suppose $m \notin N + J(M)$, since $rm \in N$, but N is a primary submodule of M and $m \notin N$, then $r^k \in [N: M] \subseteq [N + J(M): M]$ for some $k \in Z^+$.Which implies that nearly Primary submodule of M, but the converse is not true in general for example let $N = \langle \frac{1}{p^i} + Z \rangle$ be a submodule of $Zp\infty$ as Z − module, Where p is a prime number. $N \ni \frac{1}{p^i} + Z = P^k \,.\, \frac{1}{p^{i+k}} + Z$, for some $k \in Z^+$. Then $\frac{1}{p^{i+k}} + Z \notin N$ and $P^k \notin [N: Zp\infty] = 0$. Thus N is not primary submodule of $Zp\infty$, but N is nearly Primary submodule of $Zp\infty$, since $\frac{1}{p^{i+k}} + Z \in N + J(M)$, where $J(Zp\infty) = Zp\infty$.
2) If $N = \langle 4 \rangle$, then N is a nearly primary submodule of Z as Z −module, where $J(Z) = 0$, $[N: Z] = \langle 4 \rangle$
3) Let $M = Z \oplus Z$ as Z −module and consider the the submodule $N = 2Z \oplus \langle 0 \rangle$ of $Z \oplus Z$, then $[N + J(Z): Z] = 0$, hence $N \ni (4,0) = 2^2(1,0)$, which implies that $(1,0) \notin N + J(Z)$ and $2^2 \notin [N + J(Z): Z] = 0$, thus N is not nearly Primary submodule of Z as Z −module.
4) The intersection of two N. Primary submodules of an R −module M not necessary is N. Primary submodules of M for example $N_1 = \langle \bar{2} \rangle$ and $N_2 = \langle \bar{3} \rangle$ be two N. Primary submodules of Z_6 as Z −module, but $N_1 \cap N_2 = \langle \bar{0} \rangle$ is not N. Primary since $\langle \bar{0} \rangle = \bar{0} = 2.\bar{3}$ neither $\bar{3} \in N_1 \cap N_2$ nor $2^k \in [N_1 \cap N_2 + J(Z): Z] = 0$, for some $k \in Z^+$.
5) Let M_1 and M_2 be R −modules, if N_1 and N_2 are N. Primary submodules of M_1 and M_2 respectively, then it is not necessary $N_1 \oplus N_2$ is nearly Primary submodule in $M_1 \oplus M_2$, for example consider the Z −module $M = M_1 \oplus M_2$ .Let $N_1 = \langle \bar{0} \rangle$ and $N_2 = 2Z$. Then N_1 and N_2 are nearly Primary in Z_4 and Z respectively.

However $N_1 \oplus N_2$ is not nearly Primary submodule in M. Since $2^2(\bar{1},1) = (\bar{0},4) \in N_1 \oplus N_2$, but $(\bar{1},1) \notin (N_1 + N_2) + J(M)$ and $(2^2)^k \notin [(N_1 + N_2) + J(M):M] = 0$ for each $k \in Z^+$.

Compare the following with proposition (2.12)

Proposition(3.3):
If N is nearly primary submodule of an R-submodule module of M, K is a proper submodule of M such that $K \nsubseteq N$ and $J(K) = J(M)$, then $N \cap K$ is nearly primary in K .

Proof: Since $K \nsubseteq N$, then $K \cap N$ is a proper submodule in K . Let $r \in R, x \in K$ such that $rx \in N \cap K$. We want to show that either $x \in N \cap K + J(K)$ or $r^n \in [(N \cap K) + J(K):K]$, for some $n \in Z^+$. Suppose that $r^n \notin [(N \cap K) + J(K):K]$, then $r^n K \nsubseteq (N \cap K) + J(K)$, then $r^n K \nsubseteq N + J(K)$ since N is nearly primary of M, then either $x \in N + J(M)$, but $J(M) = J(K)$, then $x \in N + J(K)$ and $x \in K$, hence $rx \in \big(N + J(K)\big) \cap K$, thus $rx \in (N \cap K) + J(K)$ or $r^n \in [N + J(K):M]$, then $r^n M \subseteq N + J(K)$, we shall get contradaction, then $x \in (N \cap K) + J(K)$. Which implies that $N \cap K$ is nearly primary submodule in K.

Preposition (3.4):
Let R be a good ring and N be a nearly primary submodule of an R - module M, K be a proper submodule of M such that $K \nsubseteq N$ and, then $N \cap K$ is nearly primary in K.

Proof: Since $K \nsubseteq N$, then $K \cap N$ is a proper submodule in K. Let $r \in R, x \in K$ such that $rx \in N \cap K$. We want to show that either $x \in N \cap K + J(K)$ or $r^n \in [(N \cap K) + J(K):K]$, for some $n \in Z^+$. Suppose that $r^n \notin [(N \cap K) + J(K):K]$, then $r^n K \nsubseteq (N \cap K) + J(K)$, since $J(K) \subseteq J(M)$, then $r^n K \nsubseteq N + J(M)$. But N is nearly primary of M, then either $x \in N + J(M)$, then $x \in N + J(M)$ and $x \in K$, hence $x \in \big(N + J(M)\big) \cap K$, thus $rx \in (N \cap K)+(J(M) \cap K)$, since R is a good ring [7] , then $x \in (N \cap K)+ J(K)$, which implies that $N \cap K$ is nearly primary submodule in K

Preposition(3.5):
Let N and K be two nearly primary submodules of M and either $J(M) \subseteq N$ or $J(M) \subseteq K$, then $N \cap K$ is nearly primary of M.

Proof: Since $K \cap N \subseteq N$ and N is a nearly primary submodule of M, then $K \cap N$ is a proper submodule in M . Let $r \in R, x \in M$ such that $rx \in N \cap K$ .We want to show that either $x \in N \cap K + J(M)$ or $r^n \in [(N \cap K) + J(M):M]$, for some $n \in Z^+$. Suppose that $r^n \notin [(N \cap K) + J(M):M]$, then $r^n M \nsubseteq (N \cap K) + J(M)$, then $r^n M \nsubseteq N + J(M)$ and $r^n M \nsubseteq K + J(M)$. But N and K are two nearly primary submodules of M, then either $x \in N + J(M)$ or $r^n \in [N + J(M):M]$, for some $n \in Z^+$ and either $x \in K + J(M)$ or $r^n \in [K + J(M):M]$, for some $n \in Z^+$, then $x \in (N + J(M)) \cap (K + J(M))$. If $J(M) \subseteq N$, then $x \in N \cap \big(K + J(M)\big)$, then $x \in N \cap K + J(M)$. If $J(M) \subseteq K$, then $x \in (N + J(M))K$, hence $x \in (N \cap K) + J(M)$, which implies that $N \cap K$ is nearly primary in M.

Corollary (3.6):

If N is a maximal submodule of an R - module M and K is nearly primary submodul of M, then $N \cap K$ is nearly primary submodule of M.

Proof: Let N be a maximal submodule of M, then $J(M) \subseteq N$, and by Preposition(2.19), which implies that $N \cap K$ is nearly primary submodule of M.

If $\phi: M \longrightarrow M'$ is an epimorphism and N is a primary submodule of an R-module M containing $Kerf$, then $\phi(N)$ is a primary submodule of M', where M and M' are two R −modules [12]

Proposition (3.8):

Let $\phi: M \longrightarrow M'$ be an $R-$epimorphism. If N is a nearly primary submodule of an R-module M containing $Ker\phi$, then $\phi(N)$ is a nearly primary submodule of M', where M and M' are two $R-$modules.

Proof: $\phi(N)$ is a proper submodule of M'. Suppose not $\phi(N) = M'$, let $m \in M$ such that $\phi(m) \in M' = \phi(N)$, $\exists\, n \in N$ such that $\emptyset(n) = \emptyset(m)$ hence $\emptyset(n-m) = 0$ then $n - m \in \ker\emptyset \subseteq N$, then $m \in N$, hence $N = M$ (contradiction), since $N \subsetneq M$. Let $r \in R$, $m' \in M'$ such that $r\,m' \in \emptyset(N)$, we want to show that either $m' \in \phi(N) + J(M')$ or $r^k \in [\,\phi(N) + J(M') : M']$ for some $k \in Z^+$. Suppose $r \notin [\,\phi(N) + J(M') : M']$. Since $\emptyset$ is onto, then $\exists m \in M$ such that $\emptyset(m) = m'$, then $\emptyset(N) \ni r\,m' = r\,\emptyset(m) = \emptyset(r\,m)$, then $\exists\, y \in N$ such that $\emptyset(y) = \emptyset(r\,m)$, hence $\emptyset(r\,m - y) = 0$, then $y - r\,m \in \ker\emptyset \subseteq N$, then $r\,m \in N$, but N is nearly primary submodule in M, then either $m \in N + J(M), \emptyset(m) = \emptyset(m) \in \emptyset(N) + \emptyset\big(J(M)\big) \subseteq \emptyset(N) + J(M')$, then $\emptyset(m) = m' \in \emptyset(N) + J(M')$ or $r^k \in [N + J(M) : M]$ for some $k \in Z^+$, then $r^k M \subseteq N + J(M)$, thus $r^k m \in N + J(M)$, then $\emptyset\big(r^k m\big) \in \phi(N + J(M))$, then $r^k\,\emptyset(m) \in \phi(N) + +\emptyset\big(J(M)\big) \subseteq \phi(N) + J(M')$, then $r^k m' \in \phi(N) + J(M')$, then $r^k M' \subseteq \phi(N) + J(M')$, thus $r^k \in [\,\phi(N) + J(M') : M']$ (contradiction with assupation), then $m' \in \emptyset(N) + J(M')$. Which implies that $\emptyset(N)$ is nearlyprimary submodule of M'.

Preposition (3.9):

If M and M' are R- modules, and $\emptyset: M \to M'$ is an $R-$epimorphism, If N' is nearly semiprime in M' and $\ker\emptyset \ll M$, then $\emptyset^{-1}(N')$ is nearly primary in M.

Proof: It is clear that $\emptyset^{-1}(N') \subsetneq M$, let $r \in R$, $m \in M$ such that $r\,m \in \emptyset^{-1}(N')$, we want to show that either $m \in \emptyset^{-1}(N') + J(M)$ or $r^k \in [\emptyset^{-1}(N') + J(M) : M]$ for some $k \in Z^+$. Suppose $r^k \notin [\emptyset^{-1}(N') + J(M) : M]$. Since $r\,m \in \emptyset^{-1}(N')$, then $\emptyset(r\,m) \in N'$, hence $r\,m' = r\,\emptyset(m) \in N'$, where $m' \in M'$, but , N' is nearly primary submodule in M'. Then either $m' \in N' + J(M')$, then $m \in \emptyset^{-1}(N') + J(M)$ or $r^k \in [N' + J(M') : M']$ for some $k \in Z^+$, then $r^k M' \subseteq N' + J(M')$, thus $r^k m' \in N' + J(M')$, then $r^k \emptyset(m) \in N' + J(M')$, then $\emptyset(\,r^k m) \in N' + J(M')$, Since $\emptyset$ is an epimorphism and $\emptyset\big(J(M)\big) = J(M')$ [7] $r^k m \in \emptyset^{-1}(N') + J(M)$, then $r^k M \in \emptyset^{-1}(N') + J(M)$, then $r^k \in [\emptyset^{-1}(N') + J(M) : M]$ (contradiction with assupation), then $m' \in N' + J(M')$, which implies that $\emptyset^{-1}(N')$ is nearly primary submodule in M.

Recall that an R- module M is called fully primary if every submodule in M is primary submodule [13].

We introduce the following

Definetion(3.10):

An R- module M is called fully nearly primary if every submodule in M is nearly primary submodul.

Preposition:(3.11):

If N is a proper submodule of an $R-$module M and $\frac{M}{N}$ is fully nearly primary, then N is nearly primary sumodule in M.

Proof: Let N be a proper submodule of an R- module M, let $r \in R$, $x \in M$ such that $rx \in N$,sinse $N \subsetneq M$, then $\exists$ a natural epimorphism $\pi: M \longrightarrow \frac{M}{N}$ and $rx \in N$, then $\pi(rx) \in \pi(N) \subseteq \frac{M}{N}$ but $\frac{M}{N}$ is fully nearly primary, thus $\pi(N)$ is nearly primary submodule of $\frac{M}{N}$, $\pi(rx) \in \pi(N)$, $r\,\pi(x) \in \pi(N) + J\left(\frac{M}{N}\right)$, then either $\pi(x) \in \pi(N) + J(\pi(M))$, thus $\pi(x) \in \pi(N) + \pi(J(M))$, then $x \in N + J(M)$

Or $r \in \left[\,\pi(N) + J\left(\frac{M}{N}\right) : \frac{M}{N}\right] = [\,\pi(N) + \pi(J(M)) : \pi(M)]$, then $r\pi(M) \subseteq \pi(N) + \pi(J(M))$, thus $\pi(rM) \subseteq \pi((N) + J(M))$, then $rM \subseteq N + J(M)$, then $r \in [N + J(M) : M)]$. Then N is nearly primary submodule of M .

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