ISSN NO. 2320-5407

Journal homepage: http://www.journalijar.com

INTERNATIONAL JOURNAL OF ADVANCED RESEARCH

RESEARCH ARTICLE

From general property balance equation to shear viscosity: Application of Laplace transforms technique

Roland Nwonodi Ifeanyi

Faculty of Engineering and Technology, Federal University Wukari, Wukari, P.M.B. 1020, Nigeria (234)

Manuscript Info

Manuscript History:

Received: 14 May 2015
Final Accepted: 22 June 2015
Published Online: July 2015

Key words:

shear viscosity; Laplace transform; diffusivity equation; boundary conditions; Prandtl number

Corresponding Author

Roland Nwonodi Ifeanyi

Abstract

Differential equations have solutions which contains a number of unknown integration constants that can be found by applying appropriate boundary conditions. This rather tedious procedure was overcome by the application of Laplace transform techniques, where the unknown integration constants were evaluated during the process of solution by use of straightforward algebraic equations. In this study, the Laplace transform technique was used to solve the diffusivity equation of fluid flow. The technique was used to extract the kinematic viscosity, and then the dynamic viscosity was obtained. The Prandtl number of the fluid was used to generate the time domain of the motion in the container. The shear viscosity of water at a temperature of 293.15 K was determined to be 1.0006 cp, and this compares with the value obtainable from standard experimental procedure. From the analysis in this study, it is confirmed that the viscosity of water is a strong function of temperature.

Copy Right, IJAR, 2015,. All rights reserved

1.0 INTRODUCTION

The general property equation is one of the most widely applicable laws in nature. Many real time problems can be solved by applying this law either alone or by combining heat, mass and momentum equations. The most generalized property equation in differential form is widely given as in equation 1, (Brodkey and Hersey, 1989):

$$\frac{\partial \Psi}{\partial t} + (U.\nabla)\Psi = \Psi_G + \left(\dot{\nabla}.\delta\nabla\Psi\right) - \Psi(\nabla.U) \qquad 1$$

For an incompressible fluid with constant transport coefficient, no generation, and transient flow condition, equation (1) turns into equation (2):

$$\frac{\partial \Psi}{\partial t} + (U.\nabla)\Psi = \delta(\nabla^2 \psi) \qquad 2$$

For the case of momentum transfer, equation (2) turns into equation (3):

$$\frac{\partial u}{\partial t} + (U.\nabla)u = \upsilon(\nabla^2 u) \qquad 3$$

Equation (3) is an example of a partial differential equation whose solution can be solved by first obtaining the general form of the expression, u. This general form will contain a number of integration constants, whose values can be found by applying the appropriate boundary conditions.

Laplace transform provides the engineer and scientist with a more systematic way of solving such equations by converting the differential equations into an algebraic expression and incorporating the boundary conditions right from the beginning of the solution process. Such equations abound in practice and include, but not limited to, flow of crude oil through porous and permeable formations, diffusion of species in process flow, vibration of materials while drilling for crude oil and molecular diffusion of the motions of fluids.

The shear viscosity is a measure of the shear stress induced by an applied velocity gradient. When molecules are in an applied force field or in thermal energy fields, they experience disturbances from such fields and thus respond to the effects of the fields. Transport coefficients, e.g. shear viscosity, are defined in terms of the response of the system to these perturbations, (Danielewicz and Gyulass, 2012) .

1.1 AIM/ OBJECTIVE

The objective of this study is to solve the momentum equation for the viscosity of fluids by the application of Laplace transform techniques, and to subsequently devise a scheme for computation of viscosity at any given temperature.

1.2 STATEMENT OF THE PROBLEM

To evaluate the shear viscosity of fluids, the velocity of motion and a useful time domain are usually needed. An inappropriate value of the velocity will yield inaccurate results. To apply the Laplace transform technique on differential equations, a useful set of boundary or initial conditions are needed. The momentum equation is a complicated one because the velocity vector is composed of three (3) components in the x, y and z directions. Each component can be treated as a scalar quantity. The Laplace transform usually involves integration from zero to infinity, so it is needful to ask such questions as:

- How can an accurate value of the velocity of motion in a control volume be obtained?
- What is the appropriate length scale for the transfer of momentum in the volume?
- What sets of boundary or initial condition will yield accurate result?

These questions and more issues will be attempted to be answered in this study.

1.3 SIGNIFICANCE OF STUDY AND SCOPE

i. This study presents a new scheme which can be easily used to extract the shear viscosity of a fluid in its pure state.

ii. It shows how the viscosity of fluids can be obtained from knowledge of the general balance equation using Laplace transform technique.

iii. It is useful in the design of fluid flow through conduits.

2.0 LITERATURE

There are several methods for calculating shear viscosity. In literature, the most prominently employed ones are the relaxation time theory, (Eyring et al, 1941), (Reif, 1965) and the Chapman-Engskon method, de Groot et al (1980).

In the relaxation time theory, the flow is pictured as taking place by a unit-molecular process whereby the application of a shearing force cause a single molecule to squeeze past its neighbouring particles and move into an unoccupied hole. The theory employs the basic Newton`s shearing equation for liquids, and substitutes for a derived velocity of the molecules. It has been widely used because of its simplicity to evaluate viscosity for both hadronic and partonic matter according to (Danielewicz and Gyulassy, 2012).

The viscosity of a system can also be calculated from simulations using two standard methods, (Zwanzig, 1965):

- Linear-response-based nonequilibrium molecular dynamics (NEMD) approach, where transport coefficient is related to an imposed gradient or driving force, F, and a resulting flux, J:

  $$J = \xi F_e \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad 2$$

  Where $\xi =$ transport coefficient.

- Green-Kubo methods, using equilibrium molecular dynamics (EMD), where transport coefficient is related to the time integral of a correlation function.

According to Evans and Morris, (1998) one of the simplest exact expressions for viscosity is that of the Green-Kubo relations for the linear shear viscosity. The use of the equation requires knowledge and application of the Molecular Dynamics Simulation.

In their work on transport phenomena, Brodkey and Hershey, (1989) utilized the Laplace transform technique to provide solution to the heat transfer problem. A similar method has been adopted here in this work. They applied Laplace transform to the general balance equation along with the boundary conditions thereby transforming the partial differential equation into an ordinary differential equation. By use of appropriate boundary conditions, they evaluated the constants in the differential equation and applied the inverse transform of the solution obtained. Further mathematical manipulations were used to obtain solutions in real time domain.

3.0 METHODOLOGY

The general property balance equation has been used to obtain a one dimensional transfer in the x-direction. There was no generation term and the flow was allowed to occur as a transient state flow in a cubic container of uniform dimensions. The velocities at the boundaries of the container were taken as equal to zero due to the no slip condition. The application of the Laplace transforms technique both to the boundary conditions and the general property equation that was derived resulted to equations of the form of Bessel functions. This gave rise to a solution in the Laplace domain.

The application of Laplace inverse to this solution generated the velocity distribution which contained the complimentary error functions. Further application of the boundary condition reduced the velocity distribution to a form where the diffusivity was obtained.

The computation of results in this study has been done by use of the Excel Spreadsheet. The spreadsheet calculates the dynamic viscosity of fluids by incorporating appropriate fluid data like the density, molecular weight, specific heat capacity, thermal conductivity, and Prandtl number.

4.0 ANALYSIS AND THEORY

4.1 THEORY

The viscosity of fluids is a function of time and velocity of flow as given by the general property equation. The shear viscosity can be extracted from this equation by applying the Laplace transform technique and obtaining appropriate velocity of motion of the fluids at corresponding time scale. As a result of the complexities in real time flow, the use of dimensionless numbers can prove to be important in determining the velocity of flow.

The model for the diffusivity of fluid has been given as in equation 1:

$$\frac{\partial \Psi}{\partial t} + (U.\nabla)\Psi = \Psi_G + \left(\dot{\nabla}.\delta\nabla\Psi\right) - \Psi(\nabla.U) \qquad .2$$

For one dimensional momentum transfer in the x direction with no generation and transient flow, equation 2 becomes equation 3 as given below:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \qquad 3$$

ν = kinematic viscosity of the fluid, which can be given as below:

$$\upsilon = \frac{\mu}{\rho} \qquad 4$$

4.2 PHYSICAL MODEL

The flow of an element of fluid in a cubic container can be given as in fig 1.

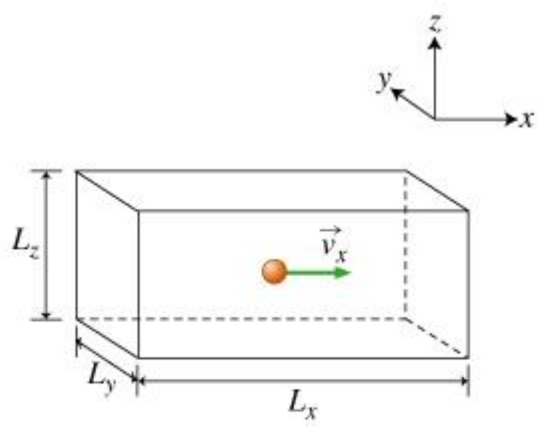

Fig 1: fluid flow through cubic container

The appropriate boundary conditions are stated as:

$$u(x=0,t)=0 \quad \text{5a}$$

$$u(x=L,t)=0 \quad \text{5b}$$

$$u(x,0)=u_o \quad \text{5c}$$

$$\frac{du}{dx}(0,t)=0 \quad \text{5d}$$

The application of Laplace transform to equation 3 is:

$$L\left\{\frac{\partial^2 u}{\partial x^2}\right\}=L\left\{\frac{1}{\nu}\frac{\partial u}{\partial t}\right\},$$

$$\frac{d^2\bar{u}}{dx^2}=\frac{1}{\nu}\left(s\bar{u}-u(x,0)\right) \quad 6$$

Invoking the boundary conditions of equations 5 and rearranging gives:

$$\frac{d^2\bar{u}}{dx^2}-\beta s\bar{u}+\beta u_o=0 \quad 7$$

$$\beta=\frac{1}{\nu} \quad 8$$

s = Laplace variable

Equation 7 which is a form of Bessel function has a solution of the form:

$$\bar{u}=\frac{u_o}{s}+C_1 Sinh(x\sqrt{s\beta})+C_2 Cosh(x\sqrt{s\beta}) \quad 9$$

Differentiation of equation 9 with respect to x gives the form below:

$$\frac{d\bar{u}}{dx}=\sqrt{s\beta}C_1 Coshx+\sqrt{s\beta}C_2 Sinhx$$

From the boundary condition, $\frac{du}{dx}(0,t)=0$

$Sinhx=0$, $Coshx=1$, thus $C_1=0$

Equation 9 becomes:

$$\bar{u} = \frac{u_o}{s} + C_2 Cosh(x\sqrt{s\beta}) \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad 10$$

Using the condition that $u(x = L, t) = 0$, the second constant can be obtained as:

$$C_2 = -\frac{u_o/s}{CoshL\sqrt{s\beta}}$$

Upon substitution, the Laplace domain solution becomes:

$$\bar{u} = \frac{u_o}{s} - \frac{u_o}{s}\left[\frac{Coshx\sqrt{s\beta}}{CoshL\sqrt{s\beta}}\right] \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad 11$$

4.3 TIME DOMAIN SOLUTION

$Cosh(x\sqrt{s\beta})$ is a hyperbolic function. For such function it holds that

$Cosha = \frac{e^{a} + e^{-a}}{2}$, thus

$$Cosh(x\sqrt{s\beta}) = \frac{1}{2}\left[e^{x\sqrt{s\beta}} + e^{-x\sqrt{s\beta}}\right] \qquad . \quad . \quad . \quad . \quad . \quad . \quad .. \qquad 12$$

Therefore, equation 11 becomes:

$$\bar{u} = \frac{u_o}{s} - \frac{u_o}{s}\left[\frac{e^{x\sqrt{s\beta}} + e^{-x\sqrt{s\beta}}}{e^{L\sqrt{s\beta}} + e^{-L\sqrt{s\beta}}}\right]. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad 13$$

As $L \to \infty$, the denominator of $\left[\frac{e^{x\sqrt{s\beta}} + e^{-x\sqrt{s\beta}}}{e^{L\sqrt{s\beta}} + e^{-L\sqrt{s\beta}}}\right] \to e^{L\sqrt{S\beta}}$

Mathematical arrangements yield the simplified form of the solution as:

$$\bar{u} = \frac{u_o}{s} - \frac{u_o}{s}\left[e^{(x-L)\sqrt{s\beta}} + e^{-(x+L)\sqrt{s\beta}}\right] \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad 14$$

The Laplace inverse of equation 14 can be taken as:

$$\bar{u} = u_o - u_o L^{-1}\left[e^{(x-L)\sqrt{s\beta}} + e^{-(x+L)\sqrt{s\beta}}\right]$$

From standard inverse integral table for exponential functions, (EqWorld, http://eqworld.ipmnet.ry):

$$\frac{e^{-\sqrt{aS}}}{S} = erfc\left(\frac{\sqrt{a}}{2\sqrt{t}}\right) \qquad 15$$

The equation for the velocity distribution becomes:

$$u = u_o - u_o\left\{erfc\frac{(L-x)}{2\sqrt{\nu t}} + erfc\frac{(L+x)}{2\sqrt{\nu t}}\right\} \qquad 16$$

An alternative form of equation 16 can be given as in equation 17:

$$\frac{u}{u_o} = 1 - \left\{erfc\frac{(L-x)}{2\sqrt{\nu t}} + erfc\frac{(L+x)}{2\sqrt{\nu t}}\right\} \qquad 17$$

4.4 LENGTH OF CONTAINER

The cubic box has a mass of fluid, m kg, flowing through it. On a molecular basis the mass of the fluid can be given as:

$$m = nM$$

$$n = \frac{N}{N_A}$$

N is the number of molecules in the box and N_A is the Avogadro`s number which equals $6.023X10^{23} molecules/mol$

Arithmetic manipulation gives the length of the box as:

$$L = \sqrt[3]{\frac{NM}{\rho N_A}} \qquad 18$$

For one mole of fluid in the box, $N = N_A$

$$L = \sqrt[3]{\frac{M}{\rho}} \qquad 19$$

M is the molecular weight of the fluid and ρ is the density

From equation 17 and the application of the boundary condition $u(x=0,t)=0$,

$$0 = 1 - \left(erfc \frac{L}{2\sqrt{\upsilon t}} + erfc \frac{L}{2\sqrt{\upsilon t}} \right) \qquad 20$$

Equation 20 simplifies to equation 21:

$$2erfc \frac{L}{2\sqrt{\upsilon t}} = 1 \qquad 22$$

$$erfc \frac{L}{2\sqrt{\upsilon t}} = 0.5 \qquad 23$$

From the table of error function, the argument that satisfies this condition lies between 0.4 and 0.5. Interpolation of data gives

$$0.4 \rightarrow 0.57161$$
$$x \rightarrow 0.5 \qquad x = 0.4778$$
$$0.5 \rightarrow 0.4795$$

Therefore the argument of the error function which satisfies equation 23 is:

$\frac{L}{2\sqrt{\upsilon t}} = 0.4778$, which can be approximated as given below:

$$\upsilon = \frac{L^2}{t} \qquad 24$$

4.5 TIME OF MOTION

The time taken for a fluid component to travel from one face of the box to the other is given as:

$$t = \frac{L}{u} \qquad 25$$

To obtain the average velocity of flow through the container, combination of momentum transfer and heat transfer is important. The equation of flow through the container (no generation and for incompressible flow), for the respective transport is:

$$\frac{\partial(\rho u)}{\partial t} + (u.\nabla)(\rho u) = \upsilon \nabla^2 (\rho u) \qquad 26$$

$$\frac{\partial(\rho c_p T)}{\partial t} + (u.\nabla)(\rho c_p T) = \alpha \nabla^2 (\rho c_p T) \qquad 27$$

From the Prandtl number, the velocity of flow can be obtained as:

$$u = \frac{k}{\rho c_p L} N_{\Pr} \quad . \quad . \quad . \quad . \quad . \quad . \quad .. \quad . \quad . \qquad .28$$

k = thermal conductivity of fluid at given temperature

c_p = specific heat capacity of fluid

$N_{\Pr}$ = Prandtl number (dimensionless)

4.6 SCHEME FOR EXTRACTION OF SHEAR VISCOSITY

- Specify fluid temperature
- With specified temperature read up fluid data: k , c_p , $N_{\Pr}$
- With fluid data calculate the average velocity through the container
- With known velocity get time of flow using equation 25
- Get the viscosity by employing equation 24

4.7 RESULTS

The viscosity of water at 293.15 K is determined in this section. At this temperature, the density is $998 Kgm^{-3}$. The result can be compared with experimental value of $0.001002\, pas - s$.

The computation is shown in the excel spreadsheet below:

DATA READ			DATA SPECIFY		
k=	0.603	W/(mK)	T=	20	deg C
Cp=	4182	J/(KgK)	M=	0.018	Kg/mol
Npr=	6.94	-	L=	0.026223	M
ρ=	998.2336361	Kg/m3			
DATA COMPUTE					
u=	3.82279E-05	m/s			
t=	685.9613544	S			
υ=	1.00245E-06	m2/s			
μ=	0.001000674	Kg/(ms)			

Symbols used in the spreadsheet are defined in the nomenclature section.

4.8 DISCUSSION

The result obtained from section 4.7 compares with that from experimental value of 0.001002 Pas-s at $20°C$. Details of the viscosity of water at different temperatures are given in table 1:

Table 1: Comparison of viscosity of Water at different temperature

Temperature, deg C	Experimental viscosity, cp	Model viscosity, cp
0	1.787	1.752
5	1.519	1.513
10	1.307	1.299
20	1.002	1.0006
30	0.7975	0.8002
40	0.652	0.6533

The excellent accuracy in the result is because the Prandtl number has been used. This number was obtained from viscosity data.

The viscosity of liquids has been said to generally decrease with increase in Temperature. An increase in temperature reduces density and causes an increase in velocity which invariably leads to reduction in viscosity of the liquid. The expression for extracting shear viscosity in this study produces results that confirm this behaviour of liquids. When the temperature increases the volume will also increase but the density will decrease, thus making the number of molecules constant. The increased velocity, due to an increase in temperature, will thus lead to a reduction in viscosity, thereby satisfying the behaviour of liquids. This can be observed as given in table 1.

5.0 CONCLUSIONS

The viscosity of water has been extracted from the general property balance equation in this study. This extraction has been made possible by the application of Laplace transform technique. From the study carried out, it is confirmed that the viscosity of liquids is a strong function of temperature.

NORMENCLATURES

L= $L_x = L_y = L_z$ = length of container, m

N = the number of molecules in the system

$\rho_L =$ Density of molecule, kg/m^3

K= thermal conductivity

Npr= Prandtl number

T= temperature of flow

U=flow velocity

t-= time of flow

V = Volume of system, m^3

μ = Dynamic viscosity of liquid, $pas - \sec$

U = velocity of particles in the system, m/s

REFERENCES

[1] Brodkey R.S., Hersey,H.C., (1989). Transport Phenomena, A unified Approach, McGraw Hill International editions, pp20-216.

[2] Danielewicz, P., Gyulassy, M. Phys.Rev.D, in: Plumari,S., Puglisi, A., Scardina, F., Greco, V., (2012). Shear viscosity of a strongly interacting system: Green-Kubo Vs. Chapman-Enskog and Relaxation Time Approximation, PhysRevC.86. arXiv:1208.0481.

[3] Eyring, H., Laidler, K.J, Glasstone, S.,(1941). The theory of rate processes, McGraw-Hill, New York.

[4] Reif, F. (1965). Fundamentals of Statistical and Thermal Physics, McGraw-Hill Book

Company, New York.

[5] De Groot, S.R., Van Leeuwen,W.A, Weert, C.,(1980). Relativistic Kinetic Theory,

Principles and Applications, North-Holland Publishing Company, Amsterdam.

[6] Zwanzig,R.,(1965). Time-correlation function & Transport coefficient in Statistical mechanics, Annu.Rev. Phys.Chem. Vol 16, pp67-102

[7] Evans,D.J., Morris,P.G.,(1998). *Transient-time correlation functions and the rheology of fluids,* Physical Review .A. 38(8), pp4142-4148.